Limits

Limit does not exist!

The Tangent and Velocity Problems

The Tangent Problem

The Tangent Problems

The word *tangent* is derived from the Latin word *tangens*, which means “touching.”

Thus a tangent to a curve is a line that touches the curve.

In other words, a tangent line should have the same direction as the curve at the point of contact.

The Tangent Problems

For a circle we could simply follow Euclid and say that a tangent is a line that intersects the circle once and only once, as in Figure 1(a).

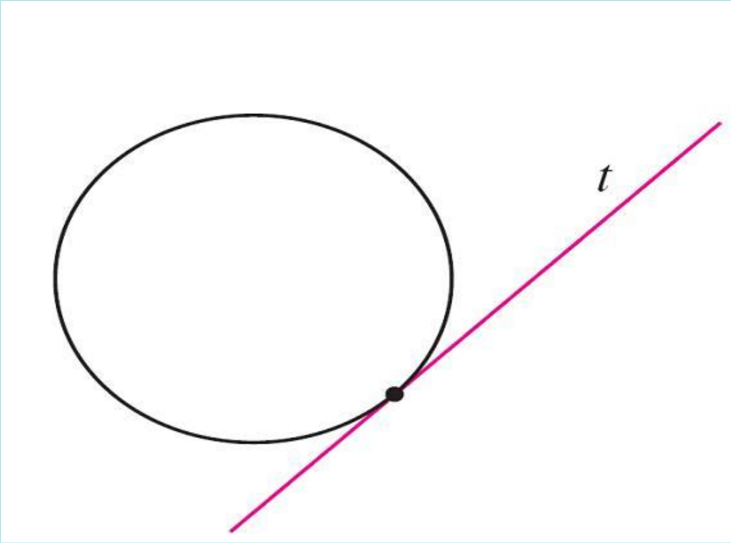


Figure 1(a)

For more complicated curves this definition is inadequate.

The Tangent Problems

Figure 1(b) shows two lines *l* and *t* passing through a point P on a curve C.

The line l intersects C only once, but it certainly does not look like what we think of as a tangent.

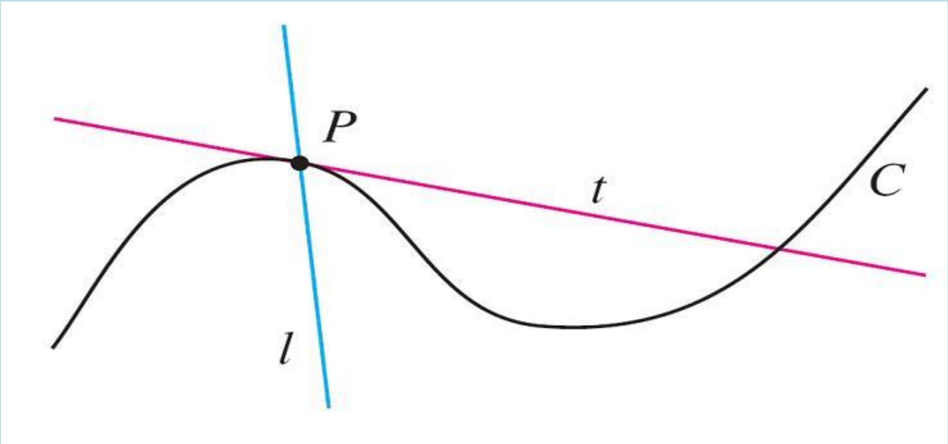


Figure 1(b)

The line *t*, on the other hand, looks like a tangent but it intersects *C* twice.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

We will be able to find an equation of the tangent line t as soon as we know its slope m.

The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope.

Example 1 – *Solution*

cont'd

But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. [A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts (intersects) a curve more than once.]

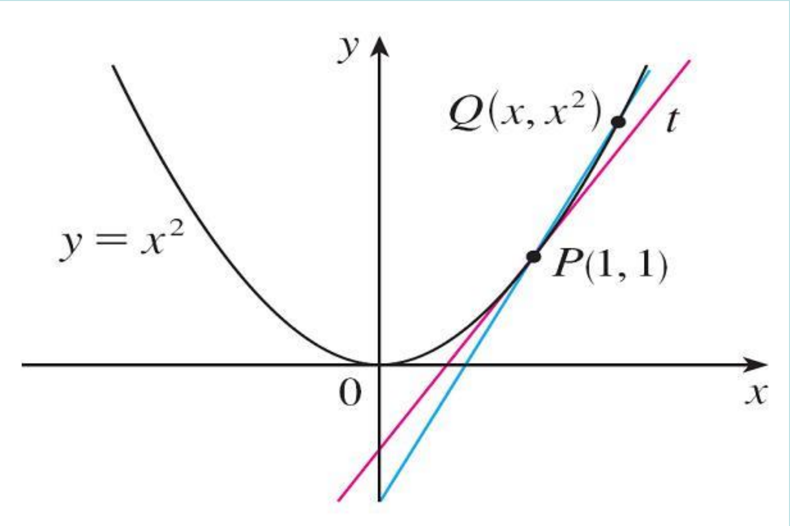


Figure 2

Example 1 – *Solution*

cont'd

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1}$$

$$= \frac{1.25}{0.5}$$

$$= 2.5$$

Example 1 – *Solution*

cont'd

The tables in the margin show the values of m_{PQ} for several values of x close to 1.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2.

Example 1 – *Solution*

cont'd

This suggests that the slope of the tangent line *t* should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

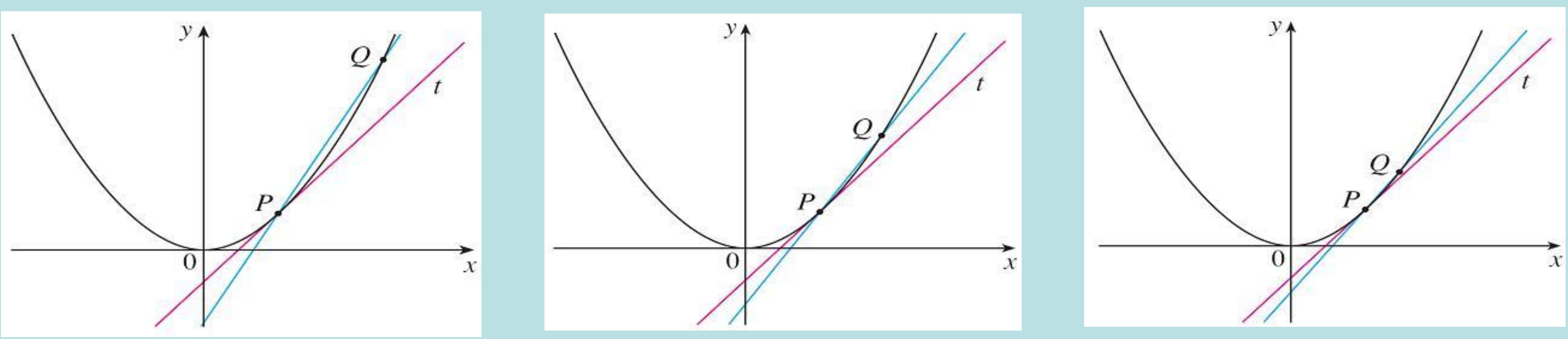
Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line $[y - y_1 = m(x - x_1)]$ to write the equation of the tangent line through (1, 1) as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

Example 1 – *Solution*

cont'd

Figure 3 illustrates the limiting process that occurs in this example.

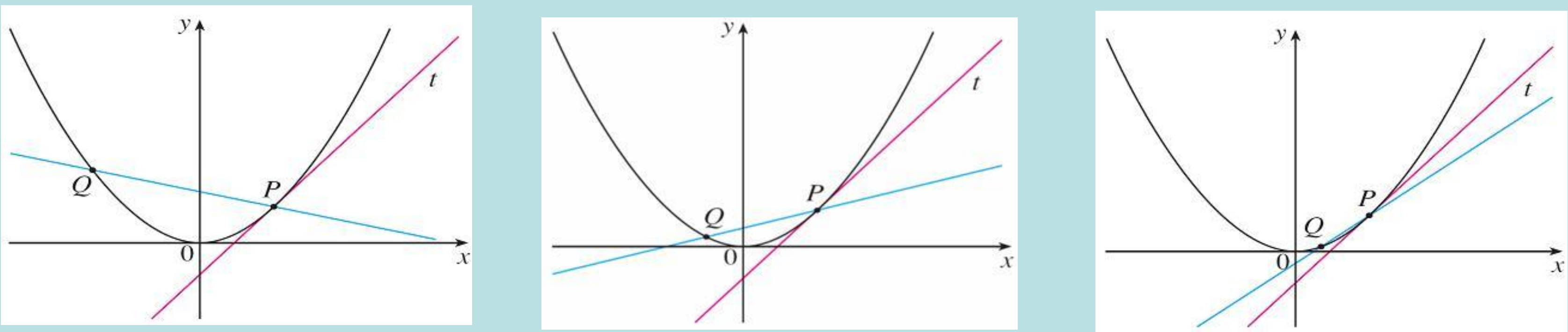


Q approaches *P* from the right

Figure 3

Example 1 – *Solution*

cont'd



Q approaches *P* from the left

Figure 3

As *Q* approaches *P* along the parabola, the corresponding secant lines rotate about *P* and approach the tangent line *t*.

The Velocity Problem

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solution:

Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.)

Example 3 – *Solution*

cont'd

If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then Galileo's law is expressed by the equation

$$s(t) = 4.9t^2$$

The difficulty in finding the velocity after 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved.

Example 3 – *Solution*

cont'd

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from t = 5 to t = 5.1:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1}$$

$$= 49.49 \text{ m/s}$$

Example 3 – *Solution*

cont'd

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 6$	53.9
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

Example 3 – *Solution*

cont'd

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s.

The **instantaneous velocity** when $t = 5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 5$.

Thus it appears that the (instantaneous) velocity after 5 seconds is

$$v = 49 \text{ m/s}$$