Limits and Derivatives

Calculating Limits Using the Limit Laws

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In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Calculating Limits Using the Limit Laws

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Calculating Limits Using the Limit Laws

Product Law

4. The limit of a product is the product of the limits.

Quotient Law

5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$ (c) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

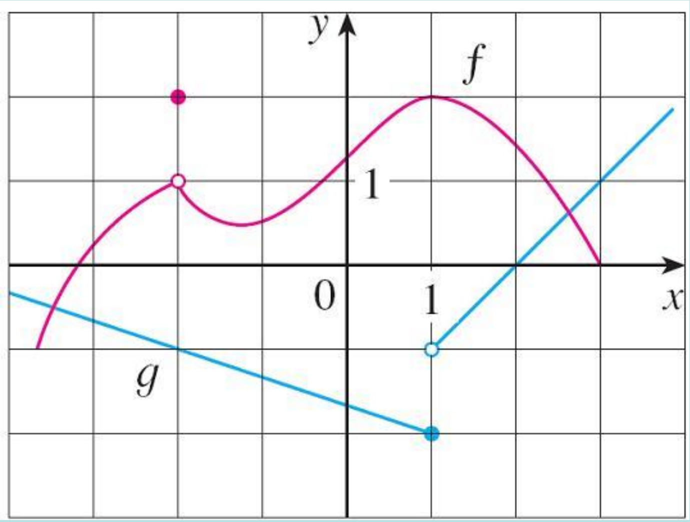

Figure 1

Example 1(a) – *Solution*

From the graphs of *f* and *g* we see that

$$\lim_{x \to -2} f(x) = 1$$ and $$\lim_{x \to -2} g(x) = -1$$

Therefore we have

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)]$$ (by Limit Law 1)

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x)$$ (by Limit Law 3)

$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – *Solution*

cont'd

We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x \to 1^-} [f(x)g(x)] = \lim_{x \to 1^-} f(x) \cdot \lim_{x \to 1^-} g(x) = 2 \cdot (-2) = -4$$

$$\lim_{x \to 1^+} [f(x)g(x)] = \lim_{x \to 1^+} f(x) \cdot \lim_{x \to 1^+} g(x) = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x \to 1} [f(x)g(x)]$ does not exist.

Example 1(c) – *Solution*

cont'd

The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5.

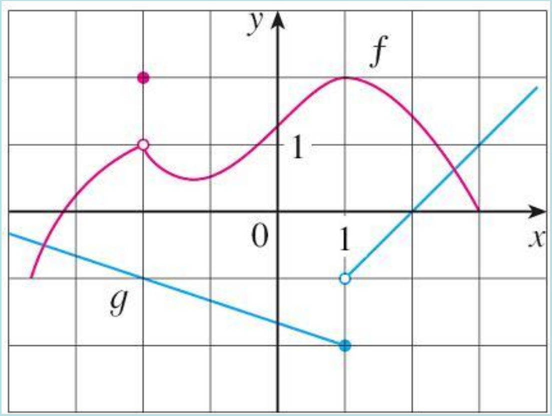

Figure 1

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

Calculating Limits Using the Limit Laws

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

In applying these six limit laws, we need to use two special limits:

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

Calculating Limits Using the Limit Laws

If we now put $f(x) = x$ in Law 6 and use Law 8, we get another useful special limit.

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

A similar limit holds for roots as follows.

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

More generally, we have the following law.

Root Law

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Calculating Limits Using the Limit Laws

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the Direct Substitution Property are called *continuous at a*.

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Calculating Limits Using the Limit Laws

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Calculating Limits Using the Limit Laws

The next two theorems give two additional properties of limits.

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Calculating Limits Using the Limit Laws

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

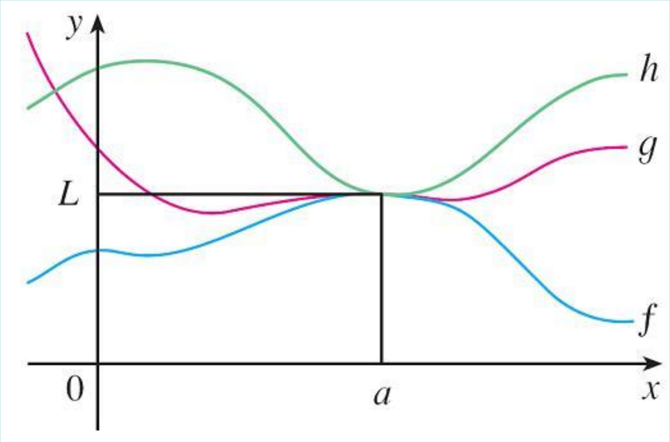

Figure 7