Limits and Derivatives

Limits at Infinity;
Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes

In this section we let *x* become arbitrarily large (positive or negative) and see what happens to *y*.

Let's begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity; Horizontal Asymptotes

The table gives values of this function correct to six decimal places, and the graph of *f* has been drawn by a computer in Figure 1.

x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

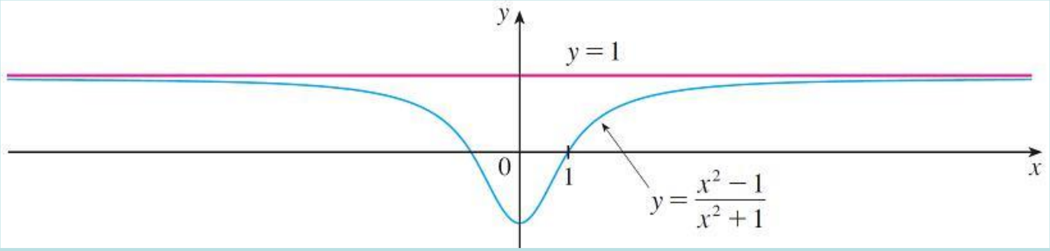


Figure 1

Limits at Infinity; Horizontal Asymptotes

As x grows larger and larger you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

$\boxed{1}$ **Intuitive Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

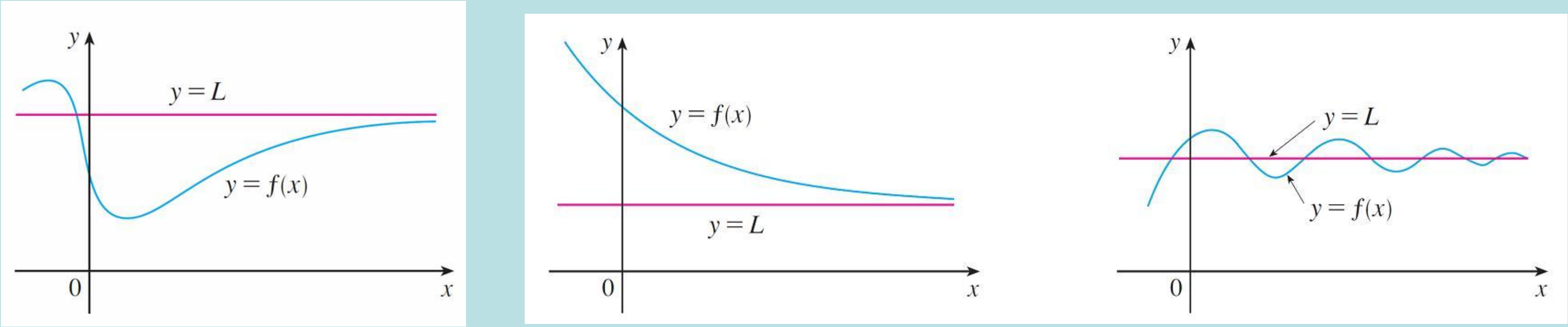
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Limits at Infinity; Horizontal Asymptotes

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Geometric illustrations of Definition 1 are shown in Figure 2.



Examples illustrating $\lim_{x \to \infty} f(x) = L$

Figure 2

Limits at Infinity; Horizontal Asymptotes

Notice that there are many ways for the graph of *f* to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

Limits at Infinity; Horizontal Asymptotes

Referring back to Figure 1, we see that for numerically large negative values of *x*, the values of *f*(*x*) are close to 1.

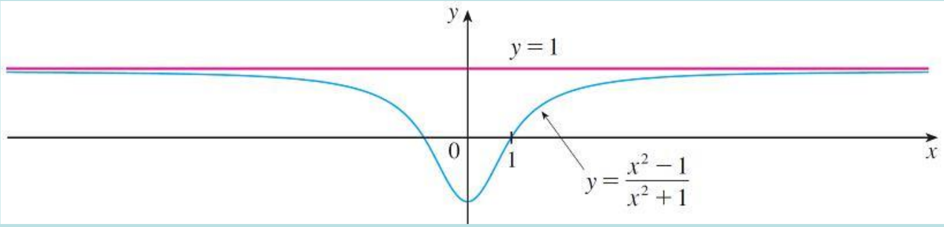


Figure 1

By letting *x* decrease through negative values without bound, we can make *f*(*x*) as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity; Horizontal Asymptotes

The general definition is as follows.

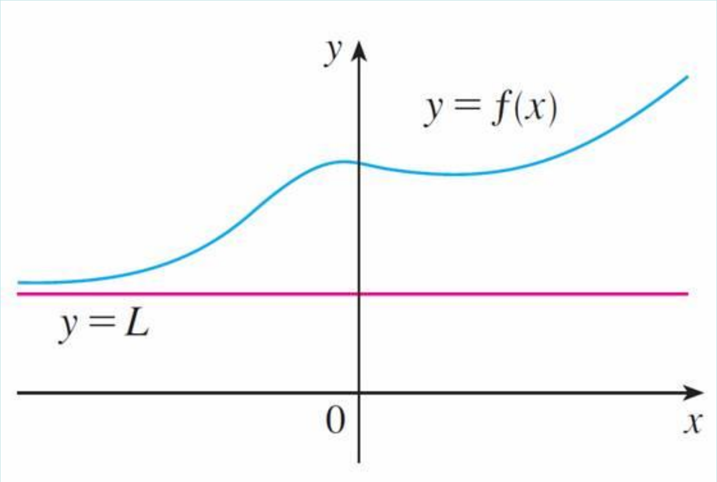
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

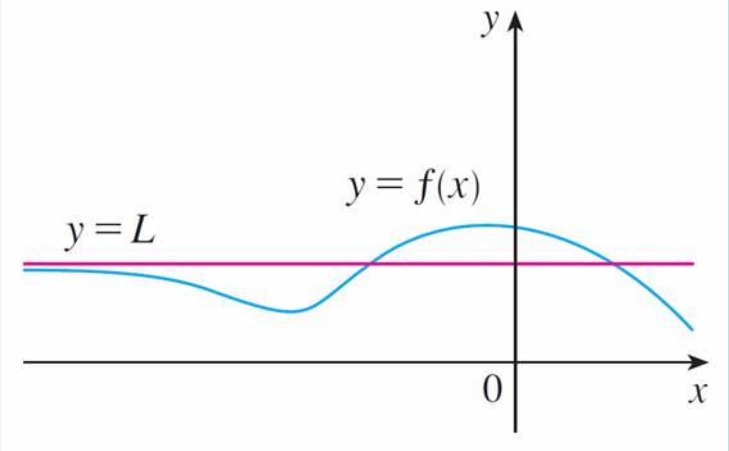
$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Limits at Infinity; Horizontal Asymptotes

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.





Examples illustrating $\lim_{x \to -\infty} f(x) = L$

Figure 3

Limits at Infinity; Horizontal Asymptotes

$\boxed{3}$ **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Example 2

Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Solution:

Observe that when *x* is large, 1/*x* is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking *x* large enough, we can make 1/*x* as close to 0 as we please.

Example 2 – *Solution*

cont'd

Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

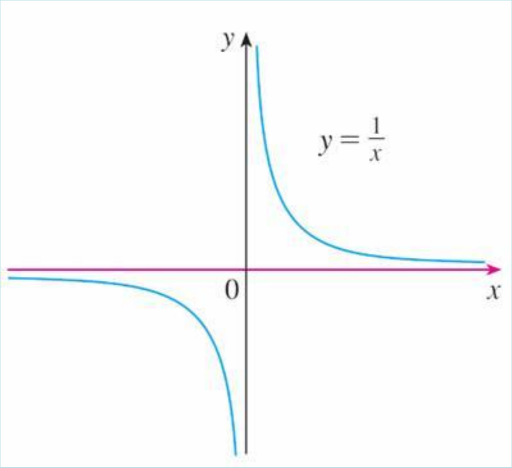
Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Example 2 – *Solution*

cont'd

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is an equilateral hyperbola; see Figure 6.)



$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Figure 6

Limits at Infinity; Horizontal Asymptotes

$\boxed{5}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3

Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ and indicate which properties of limits are used at each stage.

Solution:

As *x* becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of *x* that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of *x*.)

Example 3 – *Solution*

cont'd

In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{\dfrac{3x^2 - x - 2}{x^2}}{\dfrac{5x^2 + 4x + 1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{3 - \dfrac{1}{x} - \dfrac{2}{x^2}}{5 + \dfrac{4}{x} + \dfrac{1}{x^2}}$$

Example 3 – *Solution*

cont'd

$$= \frac{\lim_{x \to \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

(by 7 and Theorem 5)

$$= \frac{3}{5}$$

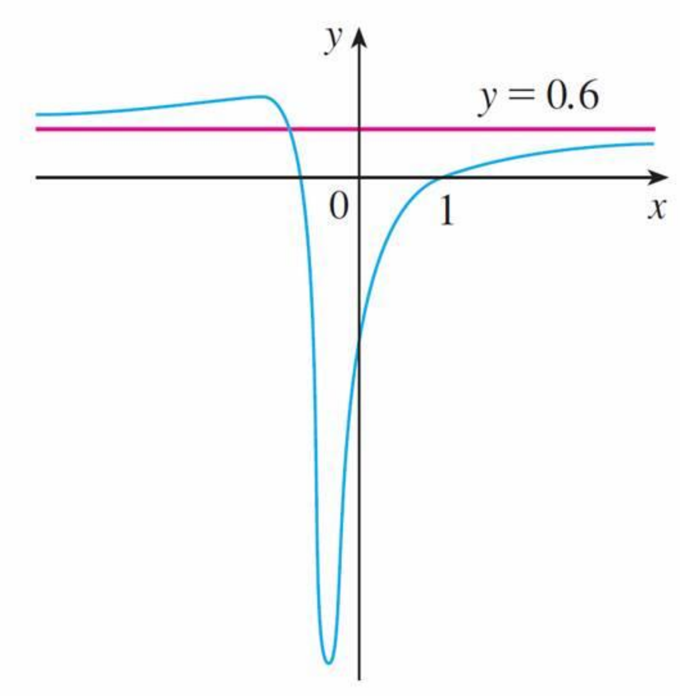
Example 3 – *Solution*

cont'd

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$.

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote

$$y = \frac{3}{5} = 0.6.$$



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Figure 7

Example 4

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Solution:

Dividing both numerator and denominator by *x* and using the properties of limits, we have

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\dfrac{\sqrt{2x^2 + 1}}{x}}{\dfrac{3x - 5}{x}}$$

Example 4 – *Solution*

cont'd

$$= \lim_{x \to \infty} \frac{\sqrt{\dfrac{2x^2 + 1}{x^2}}}{\dfrac{3x - 5}{x}}$$

(since $\sqrt{x^2} = x$ for $x > 0$)

$$= \frac{\lim\limits_{x \to \infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim\limits_{x \to \infty} \left(3 - \dfrac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim\limits_{x \to \infty} 2 + \lim\limits_{x \to \infty} \dfrac{1}{x^2}}}{\lim\limits_{x \to \infty} 3 - 5 \lim\limits_{x \to \infty} \dfrac{1}{x}}$$

$$= \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0}$$

Example 4 – *Solution*

cont'd

$$= \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of f.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

Example 4 – *Solution*

cont'd

So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2 + 1}}{x} = \frac{\sqrt{2x^2 + 1}}{-\sqrt{x^2}}$$

$$= -\sqrt{\frac{2x^2 + 1}{x^2}}$$

$$= -\sqrt{2 + \frac{1}{x^2}}$$

Example 4 – *Solution*

cont'd

Therefore

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+\dfrac{1}{x^2}}}{3-\dfrac{5}{x}}$$

$$= \frac{-\sqrt{2+\displaystyle\lim_{x\to-\infty}\frac{1}{x^2}}}{3-5\displaystyle\lim_{x\to-\infty}\frac{1}{x}}$$

$$= -\frac{\sqrt{2}}{3}$$

Example 4 – *Solution*

cont'd

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote.

A vertical asymptote is likely to occur when the denominator, $3x - 5$, is 0, that is, when $x = \frac{5}{3}$.

If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$, then the denominator is close to 0 and $3x - 5$ is positive. The numerator $\sqrt{2x^2 + 1}$ is always positive, so $f(x)$ is positive.

Therefore

$$\lim_{x \to (5/3)^+} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \infty$$

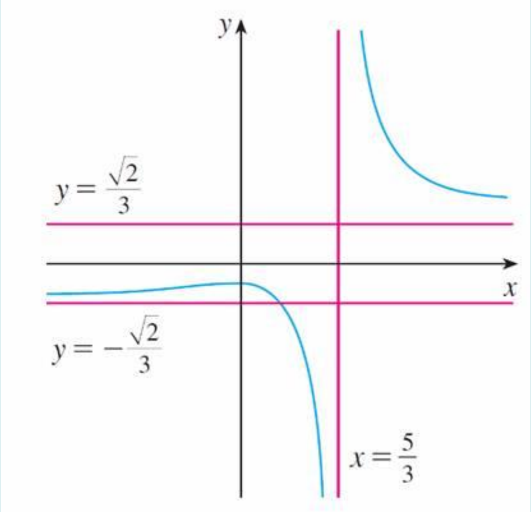
Example 4 – *Solution*

cont'd

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$, then $3x – 5 < 0$ and so $f(x)$ is large negative. Thus

$$\lim_{x \to (5/3)^-} \frac{\sqrt{2x^2 + 1}}{3x - 5} = -\infty$$

The vertical asymptote is $x = \frac{5}{3}$. All three asymptotes are shown in Figure 8.



$$y = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

Figure 8

Infinite Limits at Infinity

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of *f*(*x*) become large as *x* becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Example 9

Find $\lim_{x \to \infty} x^3$ and $\lim_{x \to -\infty} x^3$.

Solution:

When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by requiring x to be large enough. Therefore we can write

$$\lim_{x \to \infty} x^3 = \infty$$

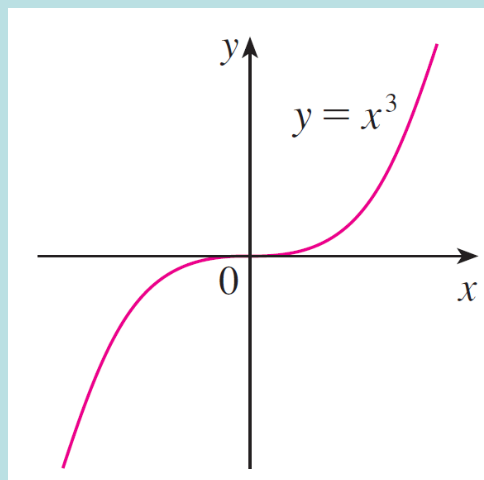
Example 9 – *Solution*

cont'd

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11.



$$\lim_{x \to \infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$$

Figure 11

Precise Definitions

Precise Definitions

Definition 1 can be stated precisely as follows.

$\boxed{7}$ **Precise Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

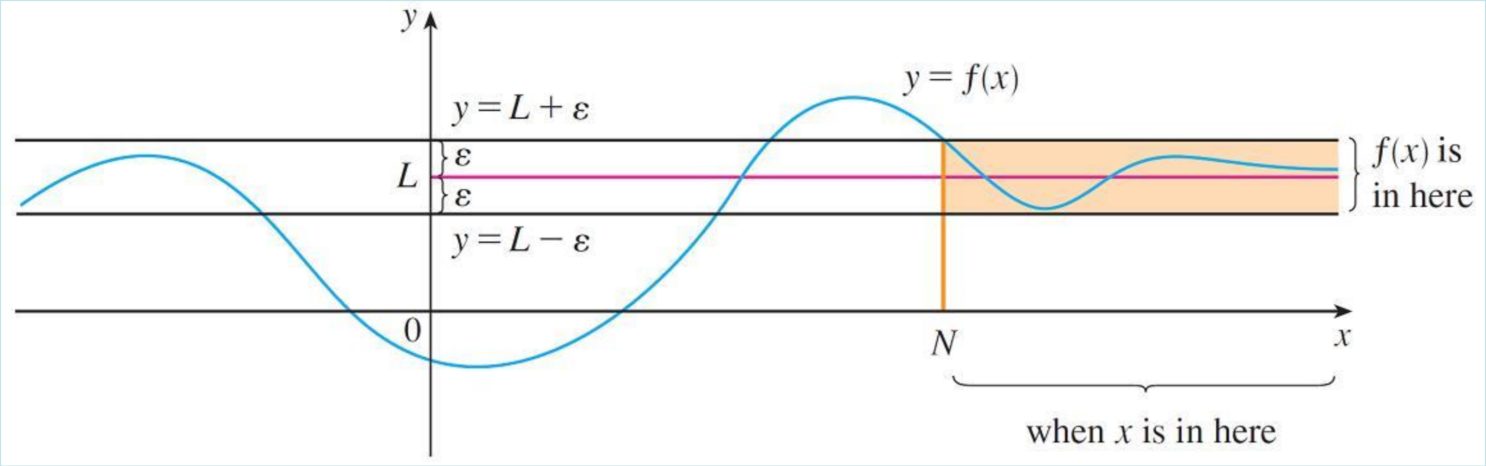
means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \qquad x > N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by requiring x to be sufficiently large (larger than N, where N depends on ε).

Precise Definitions

Graphically it says that by keeping *x* large enough (larger than some number *N*) we can make the graph of *f* lie between the given horizontal lines *y* = *L* – *ε* and *y* = *L* + *ε* as in Figure 14.

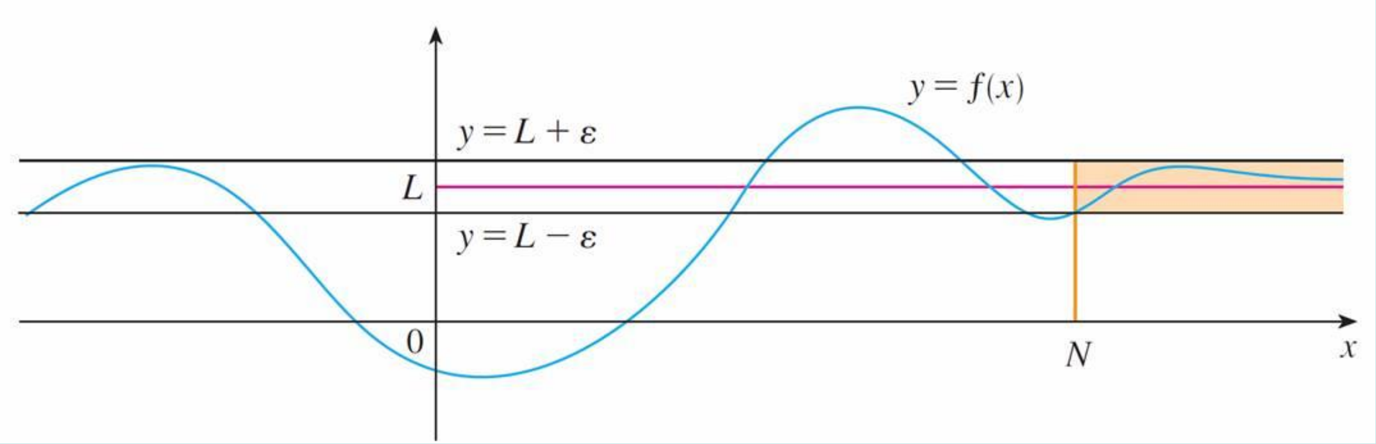


$$\lim_{x \to \infty} f(x) = L$$

Figure 14

Precise Definitions

This must be true no matter how small we choose ε. Figure 15 shows that if a smaller value of ε is chosen, then a larger value of N may be required.



$$\lim_{x \to \infty} f(x) = L$$

Figure 15

Precise Definitions

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \qquad x < N \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Example 14

Use Definition 7 to prove that $\lim_{x \to \infty} \frac{1}{x}$ = 0.

Solution:

Given $\varepsilon > 0$, we want to find N such that

if $x > N$ then $\left| \frac{1}{x} - 0 \right| < \varepsilon$

In computing the limit we may assume that $x > 0$.

Then

$1/x < \varepsilon \iff x > 1/\varepsilon$.

Example 14 – *Solution*

cont'd

Let's choose N = 1/ε. So

if $x > N = \dfrac{1}{\varepsilon}$ then $\left|\dfrac{1}{x} - 0\right| = \dfrac{1}{x} < \varepsilon$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Example 14 – *Solution*

cont'd

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

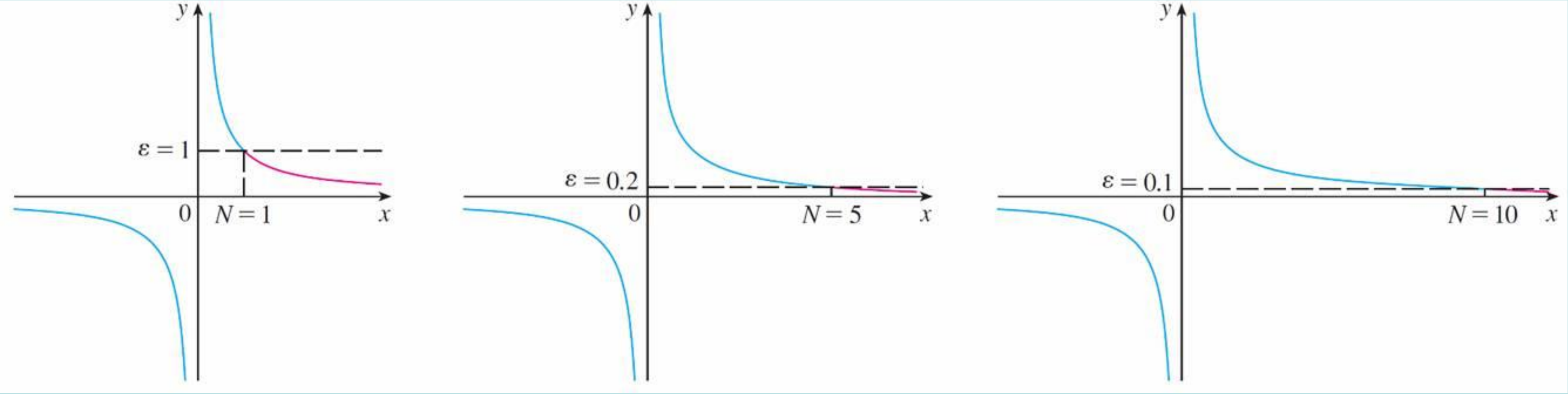
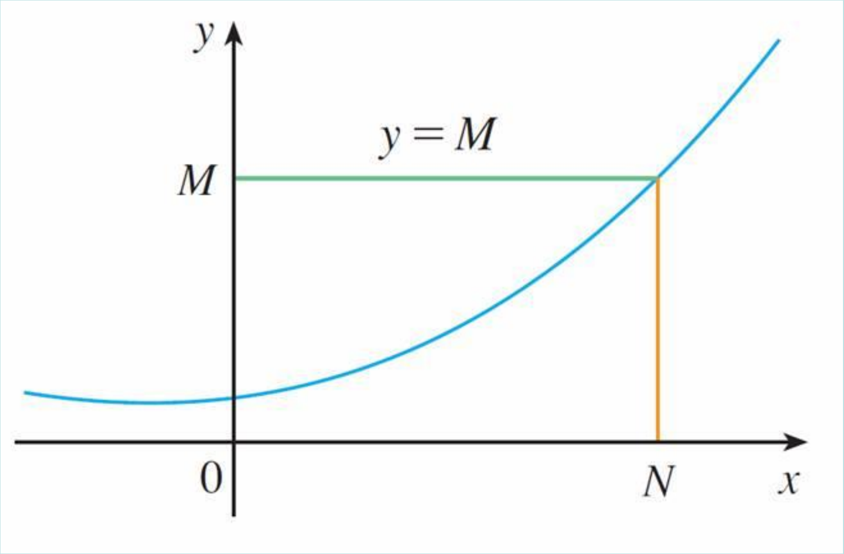


Figure 18

Precise Definitions

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.



$$\lim_{x \to \infty} f(x) = \infty$$

Figure 19

Precise Definitions

9 Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

Similar definitions apply when the symbol ∞ is replaced by $-\infty$.