Analysis of Electric Circuits Using Fourier Series

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Abstract— This work examines the analysis of electric circuit and representation of periodic functions as infinite trigonometrically series in sine and cosine terms (or complex exponentials), and presents the basic analysis of Fourier series with regard to its applications in electric circuits. A large proportion of phenomena studied in engineering and science such as alternating current and voltage are periodic in nature, and can be analyzed into their constituent components (i.e. fundamentals and harmonics) by use of Fourier series. . The electric periodic signals are decomposed into sum of a set of oscillating functions of complex exponentials.

I. INTRODUCTION

In modern technology and engineering, we usually encounter phenomena which are periodic in nature, such as alternating voltage and current. These periodic functions or signals can be analyzed into fundamentals and harmonics by means of Fourier analysis. The Fourier series, which is a branch of Fourier analysis, decomposes periodic signals into sum of infinite trigonometrically series in sine and cosine terms.
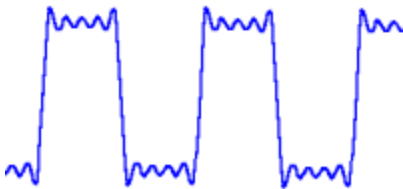

Fig. 1. A periodic function

So, we are aiming to find an approximation using trigonometric functions for various waveforms like saw tooth, square wave, etc, that occur in electronics. This is done by adding more and more trigonometric function together. Hence, Fourier series is the sum of these special trigonometric functions [5, 10, 12]. The decomposition of periodic function or periodic signal into the sum of sine and cosines was difficult until Joseph Fourier (1768 – 1830) introduced the Fourier series in his work in 1807 [1, 4]. The original motive was to solve heat problems. His work later became bedrock for solving a wide array of mathematical and physical problems, especially those involving linear differential equations with constant coefficients for which Eigen solutions are sinusoids [2]. These periodic functions can be analyzed into their harmonics and fundamentals by the process of Fourier series. Chain walled shell theory [2], Vibration analysis, Signal processing, quantum machines, acoustics, optics, econometrics [1] etc, can be solved using Fourier series. Fourier series are very vitally used to approximate a periodic waveform in electronics and electrical circuits. It is useful in mathematics as it is used extensively in calculators and computers for evaluating values of many functions [3]. Fourier series uses familiar mathematical techniques like graphs, differentiation, trigonometry, integration, summation notions; etc.It is an tool for analysis and solving problems in communication, electromagnetism, statics, quantum mechanics, etc. Generally, we use series in mathematics to approximate a complex function by a sum of simple terms. In approximating a series we normally require an infinite number of terms to completely eliminate the error between the original function and the approximation.

In Series approximation a few terms are needed for the approximation to converge sufficiently in order to give a reasonable small error. Let us consider a series expansion below:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \ldots \quad (1)$$

In the equation above, x represents the radians, the! Symbol indicates the factorial i.e., $5! = 5.4.3.2.1$. If x is small the series requires only a few terms for convergence.

II. HISTORICAL BACKGROUND

Fourier series was the brain child of Joseph Fourier (1768-1830). He was a French Mathematician taught by Lagrange and Laplace and worked as a scientist in the Napoleon's army. He made vital contributions to the theory of heat and expansion of function such as trigonometric series after preliminary investigations by Leonard Euler, Daniel Bernoulli and Jean Le Rond d'Alembert. He published his initial results in his 1807 Memoire Sur la Propagation de la Chaleur dans les Corps Solids (treatise on the propagation of heat in solid bodies) and Theories analytique de la chadeur in 1822[1,4]. Although Joseph Fourier was motivated to solve heat equation, his study later became widely used in solving array of mathematical, engineering and physical problems, especially those involving linear differential equations with constant coefficient.

III. APPLICATIONS OF FOURIER SERIES INCLUDE: [2, 3, 13]

1. Mathematical techniques like graphs, integration, differentiation, summations, notions and trigonometry.
2. Electrical electronics engineering.
3. Vibration analysis
4. Optics
5. Econometrics
6. Signal processing
7. Acoustics
8. Quantum machines
9. Thin-walled shell theory

10.   Use extensively in calculation and computers for evaluating values of many functions.

IV.   ESSENCE OF FOURIER SERIES

Fourier series has the following advantages.

1. Use for expansion of an oscillating function.
2. The non-periodic function can be expressed in Fourier series via Taylor's series.
3. Fourier series of non – periodic function is not uniformly convergent at all points.
4. There is validity in the application of term by term integration of a convergent Fourier series.
5. It is useful for mathematical analysis of electrical system.
6. It is applicable in the expansion of oscillation.

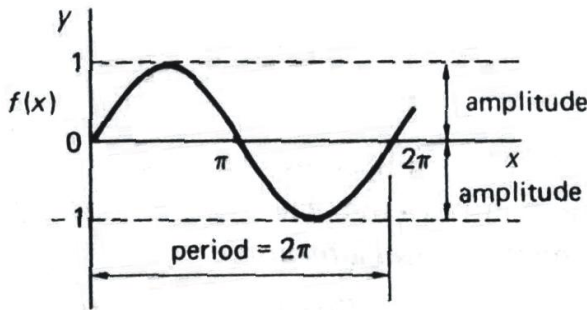## V.   SINUSOIDAL PERIODIC FUNCTION

Fig. 2 Sinusoidal Periodic Function.

If we consider the above figure, it shows a periodic function, $y = \sin x$. $f(t)$ is periodic since the function values repeat at regular interval of the independent variable. The period T, of the oscillation is the regular interval between repetitions [5,7,12]. Let a periodic function $y = \sin x$, with which x increases from 0^0 to 360^0. The period becomes 2π radians or 360^0, with maximum displacement of one 1 from the position of rest [7, 12].
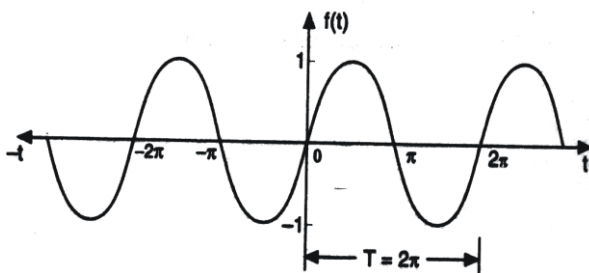

Fig.3 Sinusoidal Waveform

In the sinusoidal waveform, depicted in fig. 3..0 above, $\sin x = \sin (x + 2\pi) = \sin (x + 4\pi)$ Where $\sin x$ = Sinusoidal periodic function of 2π.

VI.   OVERVIEW OF FOURIER SERIES

In electronics, Fourier series is used to approximate a periodic waveform, in which amplitude verses time characteristic is repeated in a period, T. The periodic waveforms, viz: rectangular wave, triangular wave, sine wave, etc.Thus, Fourier series enable us to represent a periodic function as an infinite trigonometrically series in sine and cosine terms.

Generally, Fourier series can be expressed in more complex- looking expression as: [5]

$$F(t) = \frac{a_0}{2} + \sum_{n-1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right] \quad (2)$$

In equation (2), $\sum_{n-1}^{\infty}$ is a mathematical symbol that denotes an index n which is allowed to take on values of 1, 2, 3, 4 …………. up to infinity and the terms following the symbol are summed. Fourier series coefficients, a_0, a_n and b_n, which are determined by the waveform. T denotes the period of the waveform, $f(t)$ indicates that the Fourier series is to approximate function of time. The above Fourier series can be clearly expressed and the first four terms ($n = 1, 2, 3$) of the series written as:

$$f(t) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right]$$
$$+ \left(a_2 \cos\left(\frac{4\pi t}{T}\right) + b_2 \sin\left(\frac{4\pi t}{T}\right) \right)$$
$$\left[a_3 \cos\left(\frac{6\pi t}{T}\right) + b_3 \sin\left(\frac{4\pi t}{T}\right) \right] + \quad (3) \cdots$$

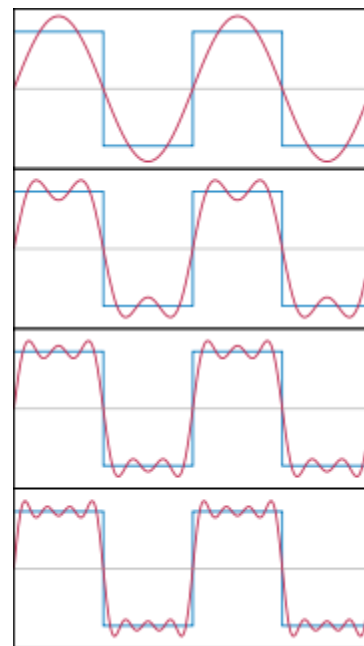The above Fourier series approximation can be depicted as in fig. 5.0.

Fig. 4 First four terms of Fourier approximation

A. PERIODIC RECTANGULAR WAVEFORM

If we consider a rectangular waveform in figure 5.0. It illustrates that A is it amplitudes, T indicate the period, ז is the pulse width. This rectangular waveform may be approximated by a Fourier series.
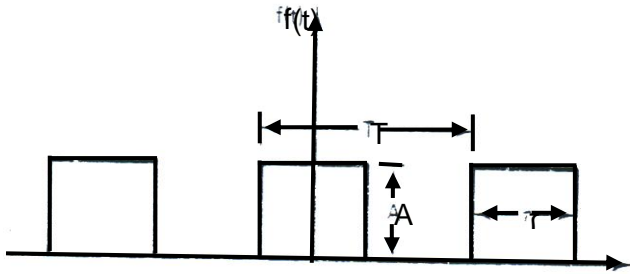

Fig. 5A Rectangular Waveform

For the rectangular waveform in fig 5.0, the Fourier coefficients are:

$$a_0 = \frac{2A\tau}{T} \quad \dots\dots\dots\dots\dots (7)$$

$$\qquad\qquad (4)$$

$$a_n = \frac{2A\tau \, Sin\left(\frac{\pi n\tau}{T}\right)}{T \left(\frac{\pi n\tau}{T}\right)} \quad \dots\dots\dots (5) \quad ..(8)$$

$b_n = 0$ (For all terms since the waveform is symmetrical about (t =0). For rectangular waveform of fig 5.0 above, the first four terms of

$$f(t) = \frac{A\tau}{T} + \frac{2A\tau}{T} \cdot \frac{Sin\left(\frac{\pi\tau}{T}\right)}{\left(\frac{\pi\tau}{T}\right)}$$

$$Cos\left(\frac{2\pi t}{T}\right) + \frac{2A}{T} \cdot \frac{Sin\left(\frac{2\pi\tau}{T}\right)}{\left(\frac{2\pi\tau}{T}\right)}$$

$$Cos\left(\frac{4\pi t}{T}\right) + \frac{2A\tau}{T} \cdot \frac{Sin\left(\frac{3\pi\tau}{T}\right)}{\left(\frac{3\pi\tau}{T}\right)}$$

$$Cos\left(\frac{6\pi t}{T}\right) + \quad \dots\dots\dots\dots\dots(9)$$

B. HALF – COSINE (OR RECTIFIED COSINE WAVEFORM)
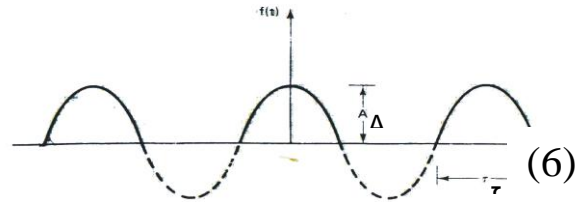

$$(6)$$

Fig. 6. Half – Cosine Waveform

This waveform is vital in communication. The Fourier coefficients for this waveform are:

$$a_0 = \frac{4A\tau}{\pi T} \quad \dots\dots\dots\dots\dots (7) \quad \dots(10)$$

$$a_n = \frac{A\tau}{T} \, Sin\left[\frac{1}{2}\left(\frac{2n\tau - 1}{T}\right)\right] +$$

$$\qquad \frac{1}{2}\left(\frac{2n\tau - 1}{T}\right)$$

$$+ \quad \frac{Sin\left[\frac{1}{2}\left(\frac{2n\tau + 1}{T}\right)\right]}{\frac{1}{2}\left(\frac{2n\tau + 1}{T}\right)} \quad -\dots\dots (8) \quad 1)$$
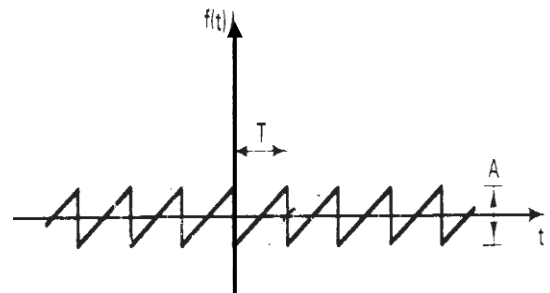
$b_n = 0$ (All terms)

C. SAW– TOOT WAVEFORM

Fig. 7. Triangular waveform

The Fourier coefficients for the above waveform are:

$a_0 = 0$
$a_1 = 0$ (for all terms)

$$b_n = \frac{-A}{n\pi} \quad \dots\dots\dots\dots\dots(12)$$

VII. CONCLUSION

Fourier series is pre-requisite for developing a solid foundation in understanding the time domain-frequency domain relationship with communication signals. It can be used to express non-sinusoidal periodic functions into a fundamental and harmonics, which can represent a function containing discontinuities. Thus, it can be conveniently use to design electrical circuits and solve problems in engineering and related fields.

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