Discrete Structures
(Discrete Mathematics)

Lecture - 1

Sets

Application of Sets

- Databases
- Data-type or type in computer programming
- Constructing discrete structures
- Finite state machine
- Modeling computing machine
- Representing computational complexity of algorithms

Set

- A **set** is an unordered collection of objects.
- The objects in a set are called the **elements**, or **members**, of the set.
- A set is said to contain its **elements**.

Example:

  - **Z** is the set of integers.
  - Cities in the Pakistan: {Lahore, Karachi, Islamabad, … }
  - Sets can contain non-related elements: {3, a, red, Gilgit }

Properties:

- Order does not matter
  - {1, 2, 3, 4, 5} is equivalent to {3, 5, 2, 4, 1}
- Sets do not have duplicate elements
  - Consider the list of students in this class
  - It does not make sense to list somebody twice

Set Membership

- **a** is an element of the set **A**, denoted by **a** $\in$ **A.**
- **a** is not an element of the set **A**, denoted by **a** $\notin$ **A.**

Sets (example)

- Example:
- **Set D**: Students taking Discrete Mathematics course.
- Assume Ali is taking Discrete Mathematics course and Saad is not taking Discrete Mathematics course.

- Ali $\in$ D
- Saad $\notin$ D

Sets (example)

- Example:

  V: {a,e,i,o,u}

  $a \in V$

  $b \notin V$

  I: {0,1,2,…,99}

  $50 \in I$

  $100 \notin I$

  S: {a,2,class}

  $2 \in S$

  $\text{room} \notin S$

Specifyinga Set

- Capital letters (A, B, S…) for sets
- Italic lower-case letter for elements (*a, x, y*…)
- Easiest way: list all the elements
  - A = {1, 2, 3, 4, 5}, Not always feasible!

- May use ellipsis (…): B = {0, 1, 2, 3, …}
- May cause confusion. C = {3, 5, 7, …}. What’s next?
- If the set is all odd integers greater than 2, it is 9
- If the set is all prime numbers greater than 2, it is 11

Set Builder

- Set builder:

  Characterize all elements in the set by stating properties they must have.

- Example:

  O= {x | x is an odd positive integer less than 10}

  O= $\{x \in Z^+ \mid x \text{ is odd and } x<10\}$

  O= {1,3,5,7,9}

  The vertical bar means “such that”

Important Sets

- Set of **natural numbers**
  - $\mathbf{N} = \{1,2,3,\ldots\}$
- Set of **integers**
  - $\mathbf{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$
- Set of **positive integers**
  - $\mathbf{Z}^{+} = \{1,2,3,\ldots\}$
- Set of **rational numbers**
  - $\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$
- Set of **real numbers**
  - $\mathbf{R}$

Examples

- S_1 = { **N, Z, Q, R** }
  - S_1 has 4 elements, each of which is a set.

- S_2 = $\{x \mid x \in \mathbf{N}$ and $\exists k \; k \in \mathbf{N}, x = k^2\}$
  - Set of squares of natural numbers

Equality of Sets

- Let A and B be two sets.
  - A and B are **equal** if and only if they have the same elements, denoted by $A = B$.

  - A and B are equal if and only if $\forall x\, (x \in A \leftrightarrow x \in B)$.

Equality of Sets (examples)

- {1,2,3} and {3,2,1}

  {1,2,3} = {3,2,1}

- $\mathbf{Z}^+$ and {0,1,2,…}

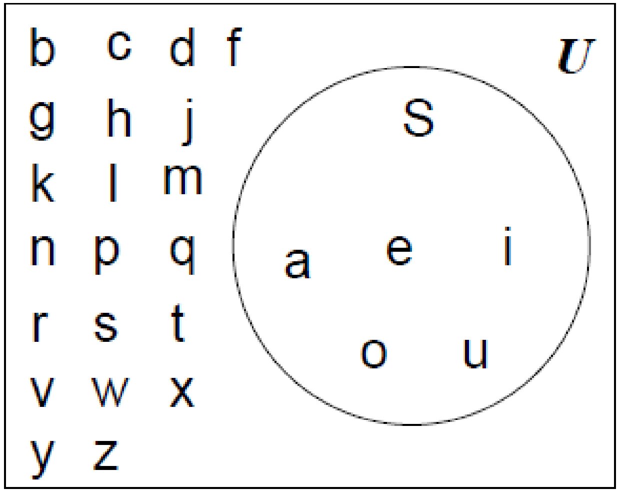
  $\mathbf{Z}^+ \neq$ {0,1,2,…}

The Universal Set

- ***U*** is the universal set – the set of all of elements (or the "universe") from which given any set is drawn
- For the set {-2, 0.4, 2}, U would be the real numbers
- For the set {0, 1, 2}, U could be the N, Z, Q, R depending on the context
- For the set of the vowels of the alphabet, U would be all the letters of the alphabet

Venn Diagrams

- Sets can be represented graphically using Venn diagram.



- The box represents the universal set
- Circles represent the set(s)
- Consider set S, which is the set of all vowels in the alphabet
- The individual elements are usually not written in a Venn diagram

Empty Set (example)

- Example:

- $S = \{x \mid x \in Z^{+} \text{ and } x < 0\}$
  $S = \{\ \} = \varnothing$

- A set that has no elements called **empty set**, or **null set**.

- $\varnothing$ and $\{\varnothing\}$
  $\varnothing \neq \{\varnothing\}$

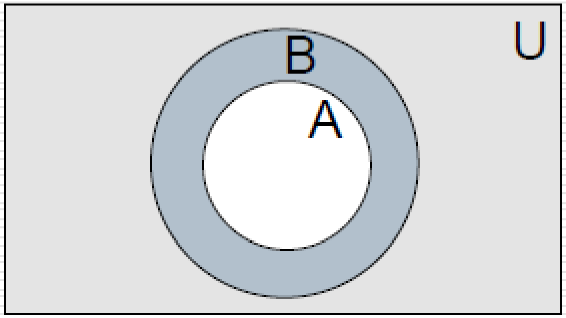
Sets Of Sets

- Sets can contain other sets
- S = { {1}, {2}, {3} }
- T = { {1}, {{2}}, {{{3}}} }
- V = { { {1}, {{2}} }, { {{3}} }, { {1}, {{2}}, {{{3}}} } }

  V has only 3 elements!
- Note that 1 ≠ {1} ≠ {{1}} ≠ {{{1}}}
- They are all different

Subset

- Let A and B be sets.
- A is a **subset** of B if and only if every element of A is also an element of B, denoted by $A \subseteq B$.
- $A \subseteq B$ if and only if $\forall x\ (x \in A \rightarrow x \in B)$.
- $\forall$ set S,
  $\emptyset \subseteq S$
  $S \subseteq S$



Subset and Equality

- $A \subseteq B$, $\forall x\ (x \in A \rightarrow x \in B)$

  and

- $B \subseteq A$, $\forall x\ (x \in B \rightarrow x \in A)$

  then

- $A = B$, $\forall x\ (x \in A \leftrightarrow x \in B)$

Subset (example)

- **Q** and **R**

  $\mathbf{Q} \subseteq \mathbf{R}$

- **N** and **Z**

  $\mathbf{N} \subseteq \mathbf{Z}$

- A = {x | x ∈ $\mathbf{Z}^+$ and x<10}

  B = {x | x ∈ $\mathbf{Z}^+$, x is even and x<10}

  B ⊆ A

Subset

- Show $\forall$ set S, $\emptyset \subseteq S$.

- Proof:

  We want to show $\forall x\ (x \in \emptyset \rightarrow x \in S)$.

- $\emptyset$ contains no element, so $x \in \emptyset$ is false.
- Hypothesis of conditional statement is false, so
  $x \in \emptyset \rightarrow x \in S$ is true.
- Thus, $\forall x\ (x \in \emptyset \rightarrow x \in S)$ is true.

Subset

- Show $\forall$ set S, $S \subseteq S$.
- Proof:

  We want to show $\forall x\ (x \in S \rightarrow x \in S)$.
- If $x \in S$ is true, then hypothesis and conclusion of conditional statement are both true and $(x \in S \rightarrow x \in S)$ is true.
- If $x \in S$ is false, then hypothesis and conclusion of conditional statement are both false and $(x \in S \rightarrow x \in S)$ is true.
- Thus, $\forall x\ (x \in S \rightarrow x \in S)$ is true.

Proper Subset

Let A and B be sets.

- A is a **proper subset** of B if and only if $A \subseteq B$ but $A \neq B$, denoted $A \subset B$.
- $A \subset B$ if and only if $\forall x\, (x \in A \rightarrow x \in B) \wedge \exists x\, (x \in B \wedge x \notin A)$.

Example

- If S is a subset of T, and S is not equal to T, then S is a proper subset of T

    Let T = {0, 1, 2, 3, 4, 5} and S = {1, 2, 3}

- S is not equal to T, and S is a subset of T

- Let Q = {4, 5, 6}. Q is neither a subset of T nor a proper subset of T

- The difference between “subset” and “proper subset” is like the difference between “less than or equal to” and “less than” for numbers

- Is $\emptyset \subseteq \{1,2,3\}$?
- Is $\emptyset \in \{1,2,3\}$?
- Is $\emptyset \subseteq \{\emptyset,1,2,3\}$?
- Is $\emptyset \in \{\emptyset,1,2,3\}$?

- Is $x \in \{x\}$
- Is $\{x\} \subseteq \{x\}$
- Is $\{x\} \in \{x,\{x\}\}$
- Is $\{x\} \subseteq \{x,\{x\}\}$
- Is $\{x\} \in \{x\}$

- Is $\emptyset \subseteq \{1,2,3\}$? Yes!
- Is $\emptyset \in \{1,2,3\}$? No!
- Is $\emptyset \subseteq \{\emptyset,1,2,3\}$? Yes!
- Is $\emptyset \in \{\emptyset,1,2,3\}$? Yes!

- Is $x \in \{x\}$
- Is $\{x\} \subseteq \{x\}$
- Is $\{x\} \in \{x,\{x\}\}$
- Is $\{x\} \subseteq \{x,\{x\}\}$
- Is $\{x\} \in \{x\}$

Size of Sets

- Let S be a set.

- The cardinality of a set is the number of elements in a set S
- **cardinality** of S, denoted by |S|.

Example

- Find cardinality of following sets.

- A = {x | x $\in \mathbf{Z}^+$, x is odd and x<10}
  A = {1,3,5,7,9}
  |A| = 5

- B = Ø
  |B| = 0

- C = {Ø}
  |C| = 1

- **R**
  R is infinite.

The Power Set

- Let S be a set.
- The **power set** of S is the set of all subsets of S, denoted by P(S).
- Example:

  P({ a ,b }) = {Ø ,{a} ,{b} ,{ a ,b }}

The Power Set (example)

- What is P({1,2,3})?
- Solution:

P({1,2,3}) = { Ø, {1}, {2}, {3}, {1,2}, {2,3}, {1,3}, {1,2,3} }

- P(Ø) = ?

- P({Ø}) = ?

The Cardinality of the Power Set

- Assume A is finite.
- $|P(A)| = ?$

Solution:

- $A = \{a\}$ $P(A) = \{\emptyset, \{a\}\}$ $|P(A)| = 2$
- $A = \{a,b\}$ $P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ $|P(A)| = 4$
- $A = \{a,b,c\}$

  $P(A)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$ $|P(A)| = 8$

- $\mathbf{|P(A)| = 2^{|A|}}$

Cartesian Product

Let A and B be sets.

- The **Cartesian product** of A and B, denoted by A x B, is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.
- $AxB = \{(a,b) \mid a \in A \wedge b \in B\}$

Cartesian Product (example)

A = {0,1,2}

B = {a,b}

Are A x B and B x A equal?

Solution:

A x B = {(0,a),(0,b),(1,a),(1,b),(2,a),(2,b)}

B x A = {(a,0),(a,1),(a,2),(b,0),(b,1),(b,2)}

So, A x B $\neq$ B x A.

The Cardinality of Cartesian Product

Assume A and B are finite.

|AxB| = ?

Solution:

- A = {a}   B={0}

  AxB = {(a,0)} |AxB| = 1

- A = {a,b}   B={0}

  AxB = {(a,0),(b,0)} |AxB| = 2

- A = {a,b}   B={0,1}   AxB={(a,0),(a,1),(b,0),(b,1)}

  |AxB| = 4

- **|AxB| = |A|.|B|**

Cartesian Product

- Let $A_1, A_2, \ldots, A_n$ be sets.
- The **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1,2,\ldots,n$.
- $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i , \forall i \in \{1,2,\ldots,n\}\}$

Cartesian Product

A = {a, b}

B = {1}

C = {x, y, z}

A x B x C = ?

Solution:

A x B x C = {(a,1,x), (a,1,y), (a,1,z), (b,1,x), (b,1,y), (b,1,z)}

The Cardinality of Cartesian product

Assume A, B and C are finite.

|AxBxC| = ?

Solution:

- A = {a} B={0} C={x} AxBxC = {(a,0,x)}
  |AxBxC| = 1
- A = {a,b} B={0} C={x} AxBxC = {(a,0,x),(b,0,x)}
  |AxB| = 2
- A = {a,b} B={0,1} C={x}
  AxB={(a,0,x),(a,1,x),(b,0,x),(b,1,x)} |AxBxC| = 4

- **|AxBxC| = |A|.|B|.|C|**

Ordered n-tuple

- The **ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its *n*th element.

- Example:

(a,b) is an ordered 2-tuple (ordered pair).

Ordered n-tuple (example)

- Assume c $\neq$ b.
- Are ordered 3-tuples (a,b,c) and (a,c,b) equal?

Solution:

- a = a but b $\neq$ c and c $\neq$ b.
- So, (a,b,c) and (a,c,b) are not equal.

Using Set Notation with Quantifiers

- $\forall x\ P(x)$ domain: S
- $\forall x \in S\ (P(x))$
- $\forall x\ (x \in S \rightarrow P(x))$

Using Set Notation with Quantifiers

- $\exists x\, P(x)$ domain: S
- $\exists x \in S\, (P(x))$
- $\exists x\, (x \in S \wedge P(x))$

Example

- What does the following statement mean?

$$\forall x \in \mathbf{R}\ (x^2 \geq 0)$$

Example

- What does the following statement mean?

$$\forall x \in \mathbf{R}\ (x^2 \geq 0)$$

Solution:

- For every real number x, $(x^2 \geq 0)$.
- The square of every real number is nonnegative.

Example

- What does the following statement mean?

$$\exists x \in \mathbf{Z}\ (x^2 = 1)$$

Example

- What does the following statement mean?

$$\exists x \in \mathbf{Z}\ (x^2 = 1)$$

Solution:

- There is an integer x such that $x^2 = 1$.
- There is an integer whose square is 1.

Truth Sets of Predicates

- Let P be a predicate and D is a domain.
- The **truth set** of P is the set of elements x in D for which P(x) is true.
- The truth set of P(x) is $\{x \in D \mid P(x)\}$.

Example

- Let $P(x)$ be $|x| = 1$ where the domain is the set of integers. What is the truth set of $P(x)$?

Example

- Let P(x) be |x| = 1 where the domain is the set of integers. What is the truth set of P(x)?

Solution:

The truth set of P(x) is {-1,1}.

Example

- Let $R(x)$ be $|x| = x$ where the domain is the set of integers. What is the truth set of $R(x)$?

Example

- Let R(x) be |x| = x where the domain is the set of integers. What is the truth set of R(x)?

Solution:

The truth set of R(x) is $x \geq 0$.

Example

- Let Q(x) be $x^2 = 2$ where the domain is the set of integers. What is the truth set of Q(x)?

Example

- Let Q(x) be $x^2 = 2$ where the domain is the set of integers. What is the truth set of Q(x)?

Solution:

The truth set of Q(x) is ∅.

Truth Set of Quantifiers

- $\forall x\, P(x)$ is true over the domain D if and only if the truth set of P is the set D.
- $\exists x\, P(x)$ is true over the domain D if and only if the truth set of P is nonempty.

Exercise Questions

Chapter # 2

Topic # 2.1

Question # 1,3,5,6,7,9,12,19,20,23,32,43,44