Discrete Structures
(Discrete Mathematics)

Lecture - 2

Set Operations

Set Operations

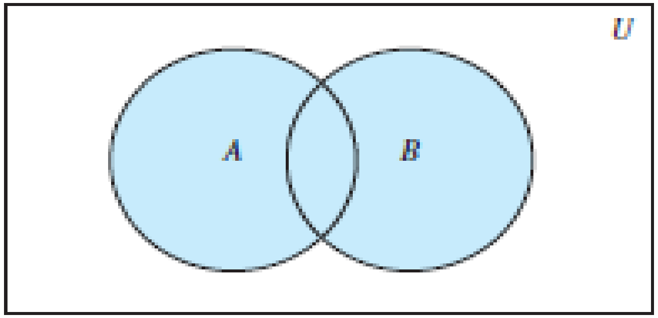
- Two sets can be combined in many different ways.
- Set operations can be used to combine sets.

Union

- Let A and B be sets.

- The **union** of A and B, denoted by **A ∪ B**, is the set containing those elements that are either in A or in B, or in both.

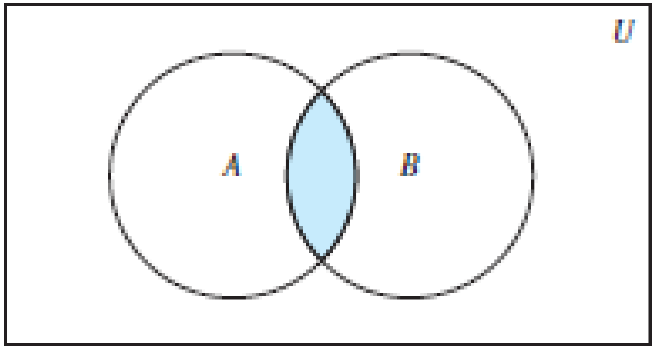
- $A \cup B = \{x \mid x \in A \vee x \in B\}$



$A \cup B$ is shaded.

Intersection

- Let A and B be sets.
- The **intersection** of A and B, denoted by **A ∩ B**, is the set containing those elements in both A and B.
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$



A ∩ B is shaded.

Union (example)

- Let $A = \{1,2,3\}$

  $B = \{2,4,6,8\}$

  $A \cup B = \{1,2,3,4,6,8\}$

- Let $A = \{x \mid x \in \mathbf{Z} \wedge x \text{ is even}\}$

  $B = \{x \mid x \in \mathbf{Z} \wedge x \text{ is odd}\}$

  $A \cup B = \mathbf{Z}$

Intersection (example)

- Let $A = \{1,2,3\}$

  $B = \{2,4,6,8\}$

  $A \cap B = \{ 2 \}$

- Let $A = \mathbf{Z}$

  $B = \{x \mid x \in \mathbf{Z} \wedge x \text{ is odd}\}$

  $A \cap B = \{x \mid x \in \mathbf{Z} \wedge x \text{ is odd}\}$

Disjoint Sets

- Two sets are called **disjoint** if their intersection is empty.
- Let $A = \{x \mid x \in \mathbf{Z} \wedge x \text{ is even}\}$

  $B = \{x \mid x \in \mathbf{Z} \wedge x \text{ is odd}\}$

  $A \cap B = \emptyset$

The Cardinality of the Union of Sets

- $|A \cup B| = ?$

Solution:

- Let $A = \{1,2,3\}$

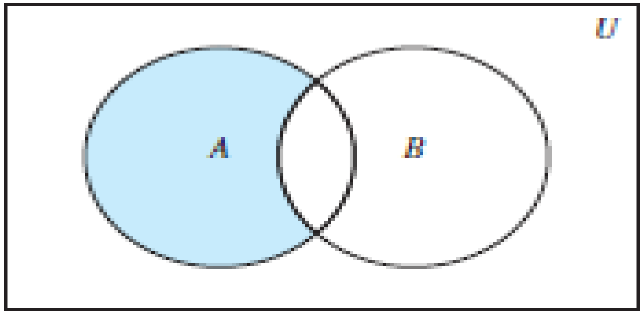
  $B = \{2,3,4\}$

  $A \cup B = \{1,2,3,4\}$
- $|A| = 3$ $|B| = 3$ $|A \cup B| = 4$

- $\mathbf{|A \cup B| = |A| + |B| - |A \cap B|}$

Difference

- Let A and B be sets.
- The **difference** of A and B, denoted by **A - B**, is the set containing those elements that are in A but not in B. (also called **complement of B with respect to A**).
- $A - B = \{x \mid x \in A \wedge x \notin B\}$



A − *B* is shaded.

Difference (example)

- Let $A = \{1,2,3\}$

  $B = \{2,4\}$

  $A - B = \{1,3\}$

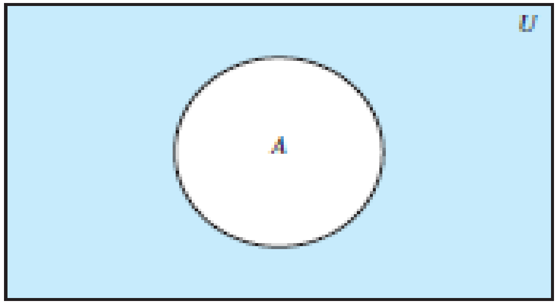
- Let $A = \mathbf{Z}$

  $B = \{ x \mid x \in \mathbf{Z} \wedge x \text{ is odd} \}$

  $A - B = \{ x \mid x \in \mathbf{Z} \wedge x \text{ is even} \}$

Complement

- Let U be the universal set and A be a set.
- The **complement** of A, denoted by A, is the complement of A with respect to U (which is U - A).
- $A = \{x \mid x \notin A\}$



$\overline{A}$ is shaded.

Complement (example)

- Let A = { a, b, c, d } and

  U is the set of English alphabet

  A = { e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }

- Let A = { x | x ∈ **Z** ∧ x is odd } and

  U is **Z**

  A = { x | x ∈ **Z** ∧ x is even }

Summary Set Operations

Operation	Notation
Union	$A \cup B = \{x \mid x \in A \lor x \in B\}$
Intersection	$A \cap B = \{x \mid x \in A \land x \in B\}$
Difference	$A - B = \{x \mid x \in A \land x \notin B\}$
Complement (U - A)	$A = \{x \mid x \notin A +$

Set Identities

$A \cup \emptyset = A$ $A \cap U = A$	Identity Laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination Laws
$A \cup A = A$ $A \cap A = A$	Idempotent Laws
$\overline{(A)} = A$	Complementation Law
$A \cup A = U$ $A \cap A = \emptyset$	Complement Laws

Set Identities

$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws

Set Identities

De Morgan's Law

$$\overline{A \cup B} = A \cap B$$

$$\overline{A \cap B} = A \cup B$$

Distributive Law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

How to Prove a Set Identity

- Four methods:
  - Use the basic set identities
  - Use membership tables
  - Prove each set is a subset of each other
  - Use set builder notation and logical equivalences

Set Identities (example)

- Show $\overline{A \cup (B \cap C)} = (C \cup B) \cap A$

Solution:

$\overline{A \cup (B \cap C)}$

$= A \cap (\overline{B \cap C})$ (By DeMorgan's Law)

$= A \cap (B \cup C)$ (By DeMorgan's Law)

$= A \cap (C \cup B)$ (By CommutativeLaw)

$= (C \cup B) \cap A$ (By Commutative Law)

What is a membership table

- Membership tables show all the combinations of sets an element can belong to
  - 1 means the element belongs, 0 means it does not
- Consider the following membership table:

A	B	$A \cup B$	$A \cap B$	A - B
1	1	1	1	0
1	0	1	0	1
0	1	1	0	0
0	0	0	0	0

Membership Table

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive Law

A	B	C	B ∩ C	A ∪ (B ∩ C)	A ∪ B	A ∪ C	(A ∪ B) ∩ (A ∪ C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Proof by showing each set is a subset of the other

- Assume that an element is a member of one of the identities
  - Then show it is a member of the other

Example

- Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

Part 1: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$Assume\ x \in A \cap (B \cup C)$

$(x \in A) \wedge (x \in (B \cup C))$	Definition of Intersection
$(x \in A) \wedge (x \in B \vee x \in C)$	Definition of Union
$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$	Distributive Law
$(x \in (A \cap B)) \vee (x \in (A \cap C))$	Definition of Intersection
$x \in (A \cap B) \cup (A \cap C)$	Definition of Union

$So,\quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Example

- Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

Part 2: $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Assume $x \in (A \cap B) \cup (A \cap C)$

$(x \in A \cap B) \lor (x \in (A \cap C))$	Definition of Union
$(x \in A \land x \in B) \lor (x \in A \land x \in C)$	Definition of Intersection
$x \in A \land (x \in B \lor x \in C)$	Distributive Law
$x \in A \land (x \in B \cup C)$	Definition of Union
$x \in A \cap (B \cup C)$	Definition of Intersection

So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof by set builder notation and logical equivalences

- First, translate both sides of the set identity into set builder notation
- Then use one side (or both) to make it identical to the other
  - Do this using logical equivalences

Set Builder Notation and Logical Equivalences (Example)

- Show $\overline{A \cap B} = A \cup B$

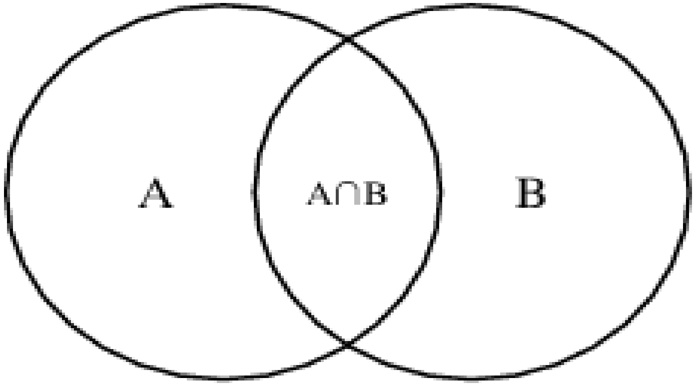
Solution:

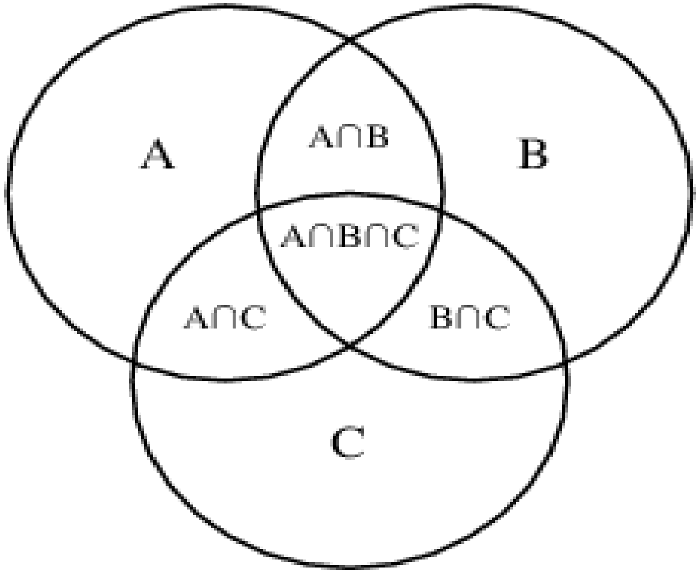
$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	Definition of Complement
$= \{x \mid \neg(x \in A \cap B)\}$	Definition of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	Definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	DeMorgan's Law
$= \{x \mid (x \notin A) \vee (x \notin B)\}$	Definition of does not belong symbol
$= \{x \mid (x \in A) \vee (x \in B)\}$	Definition of Complement
$= \{x \mid x \in A \cup B\}$	Definition of Union
$= A \cup B$	By meaning of Set Builder Notation

2 and 3 - Set Venn Diagram

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$





Example

Suppose a list *A* contains the 30 students in a mathematics class, and a list *B* contains the 35 students in an English class, and suppose there are 20 names on both lists.

Find the number of students:

- only on list *A*
- only on list *B*
- on list *A* or *B* (or both),
- on exactly one list.

Example

Solution:

- $30 - 20 = 10$ names are only on list A.
- $35 - 20 = 15$ are only on list B.
- $|A \cup B| = |A| + |B| - |A \cap B| = 30 + 35 - 20 = 45.$
- $10 + 15 = 25$ names are only on one list; that is,

  $|A \oplus B| = 25$.

Example

Consider the following data for 120 mathematics students at a college concerning the languages French, German, and Russian:

65 study French, 45 study German,

42 study Russian , 20 study French and German,

25 study French and Russian,

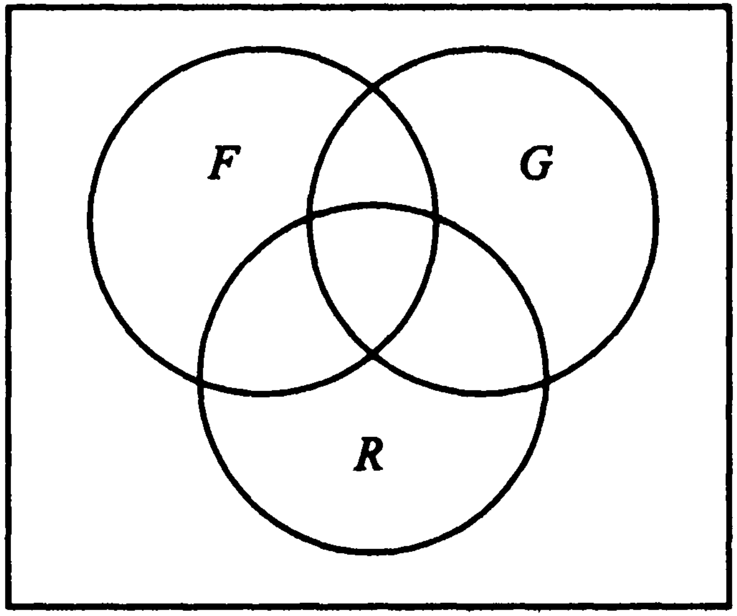
15 study German and Russian.

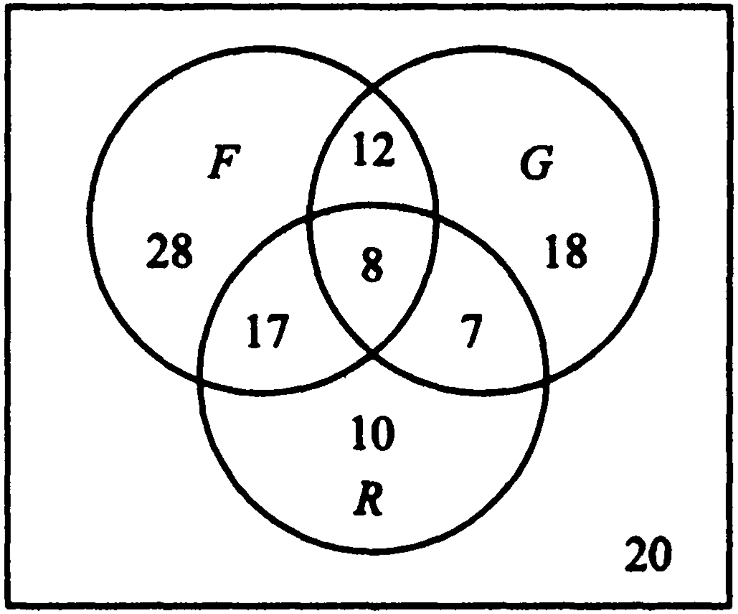
8 study all three languages.

Determine how many students study exactly 1 subject and fill the correct numbers of students in each eight region of Venn diagram shown in figure.

Example

- Total number of students exactly registered in one course = 28+18+10=56





Exercise Questions

Chapter # 2

Topic # 2.2

Question # 1, 2, 3,4,5,6,15,16,17,18,
19,20,21,22,23,24,25