# Discrete Structures
# (Discrete Mathematics)

Lecture – 3

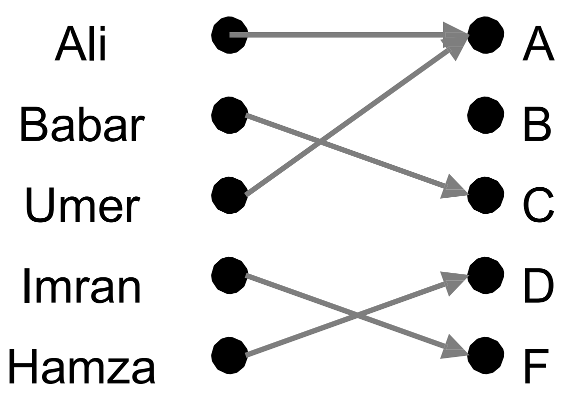
## Functions

# Application of Functions

- Define discrete structures such as sequences and strings
- Represent the time that a computer takes to solve problems of a given size
- Represent the complexity of algorithms
- …

# Functions

- In many examples we assign to each element of a set, a particular element of a second set (which may be the same as the first).
- For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A,B,C,D, F}.
- This assignment is an example of a function.



# Functions

- Let **A** and **B** be nonempty sets.
- A **function** *f* from **A** to **B** is an assignment of exactly one element of **B** to each element of **A**.
- If **b** is the unique element of **B** assigned by the function ***f*** to the element **a** of **A**, we write ***f*(a) = b**.
- If ***f*** is a function from **A** to **B**, we write ***f*: A** → **B**.

# Functions

- If ***f*** is a function from **A** to **B**, we say that **A** is the domain of ***f*** and **B** is the codomain of ***f***.

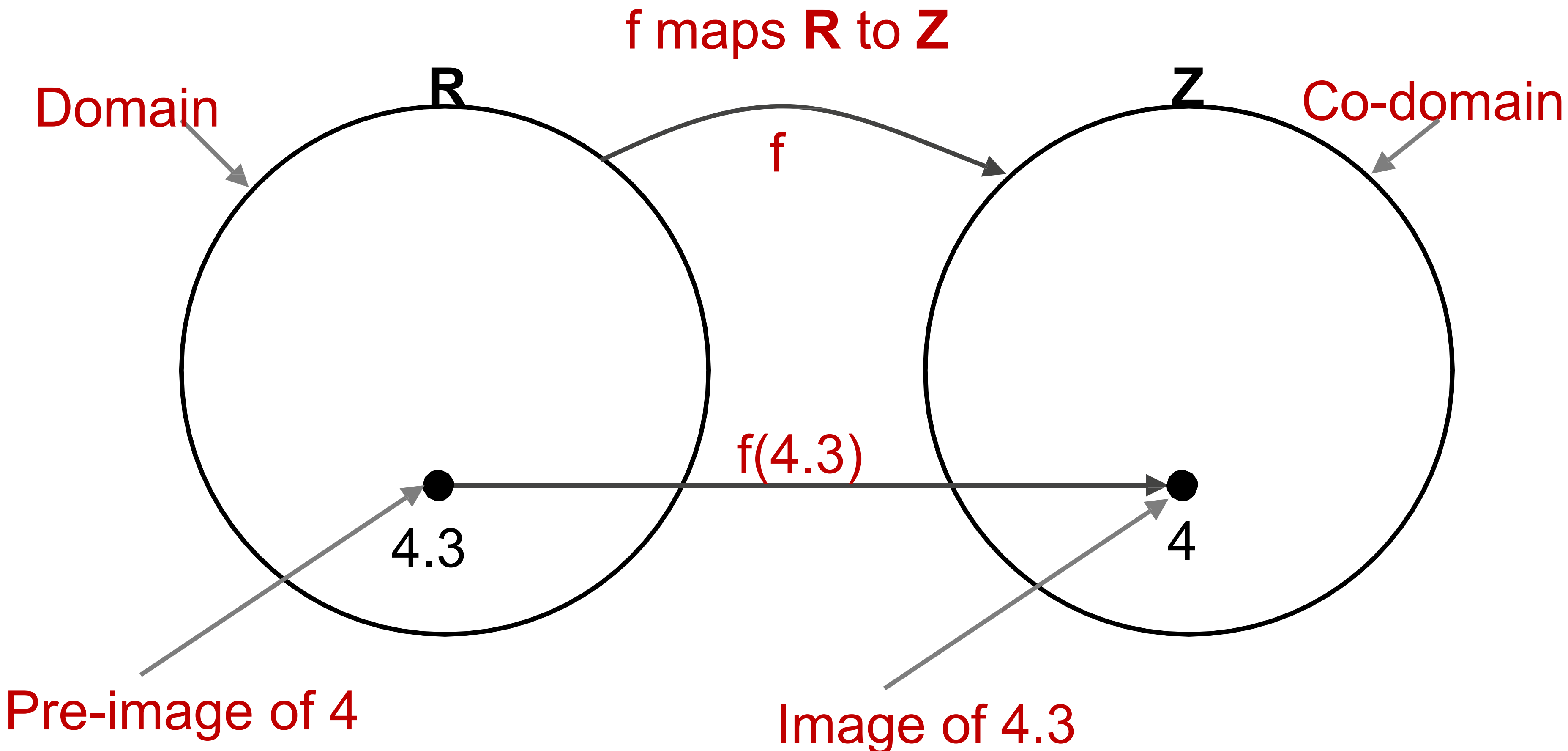
- if ***f*****(a) = b**, we say that **b** is the image of **a** and **a** is the pre-image of **b**.

- The range of ***f*** is the set of all images of elements of **A**.

- Also, if ***f*** is function from **A** to **B**, we say that ***f*** maps **A** to **B**.

# Functions

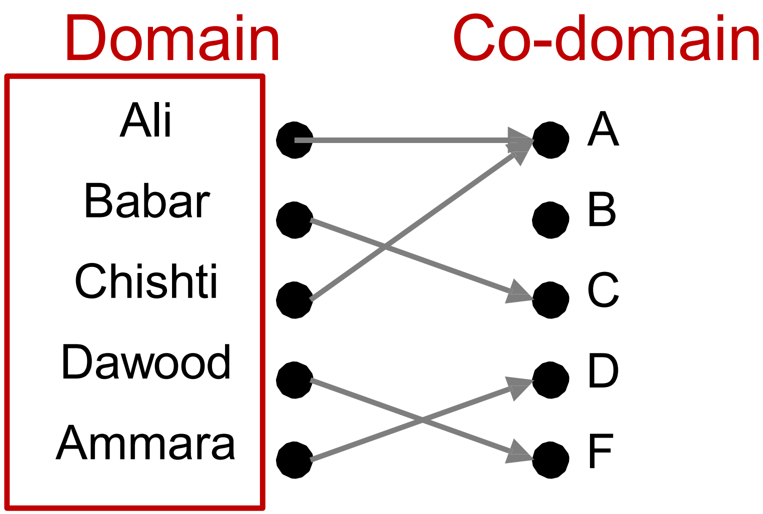
- A function takes an element from a set and maps it to a UNIQUE element in another set.

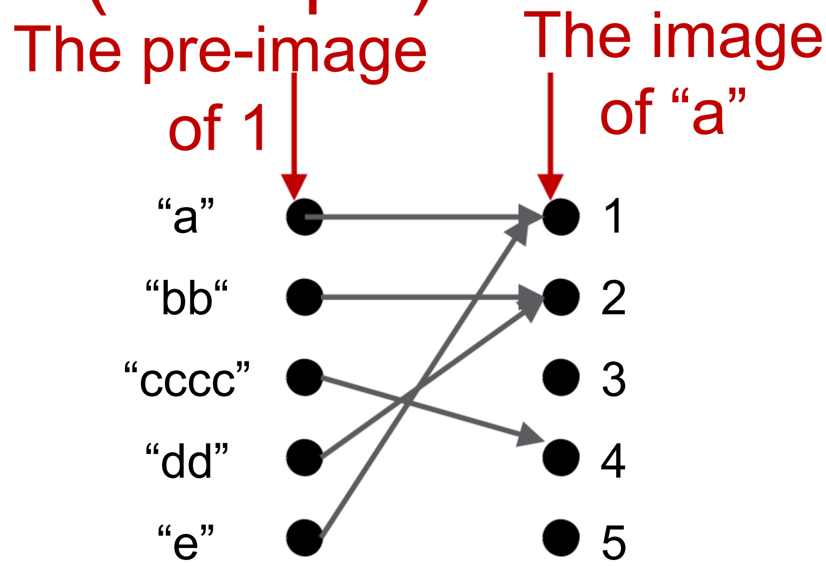


# Arrow Diagram of Functions

- The definition of a function implies that the arrow diagram for a function f has the following two properties:
- Every element of **A** has an arrow coming out of it
- No elements of **A** has two arrows coming out of it that point to two different elements of **B**.

# Arrow Diagram of Functions(example)




**A class grade function**

**g(Ali) = A**
**g(Babar) = C**
**g(Chishti) = A**
**…**

**A string length function**

**f(x) = length x**

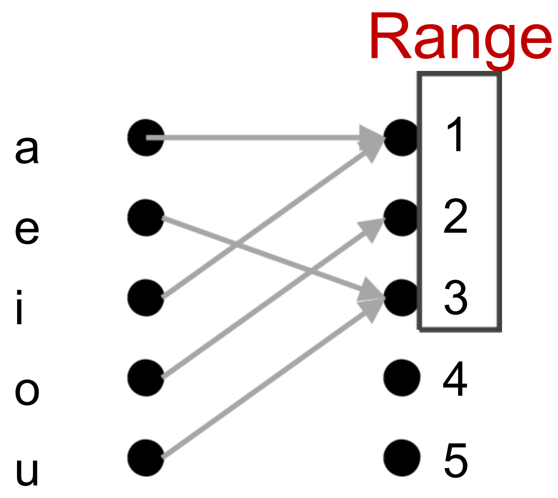
**f(“a”) = 1**

**f(“bb”) = 2**

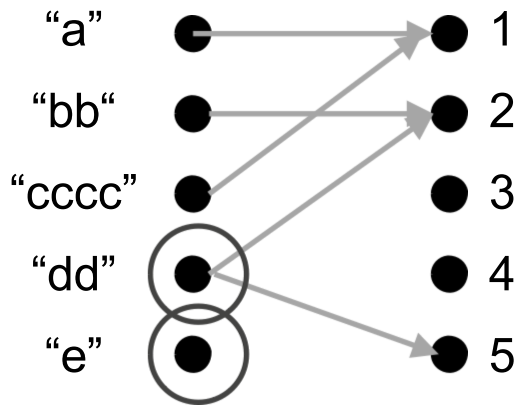
**…**

# Functions

The ***range*** of *f* is the set of all images of elements of *A*.



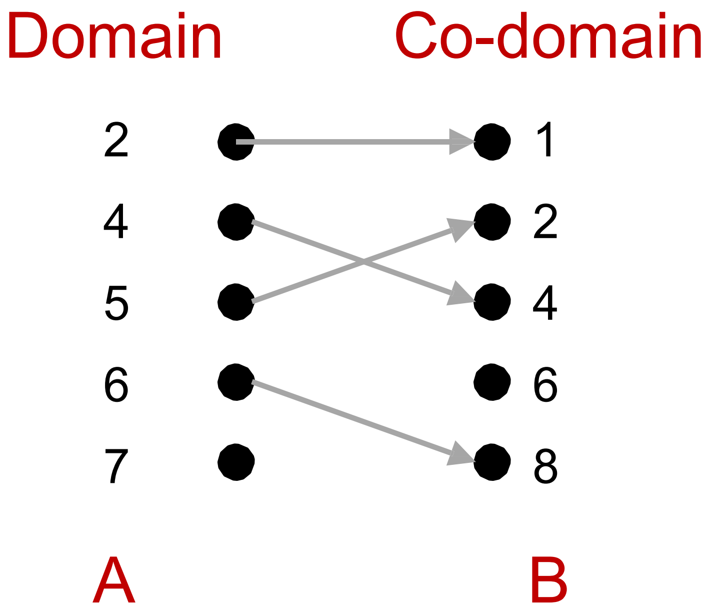
# Functions



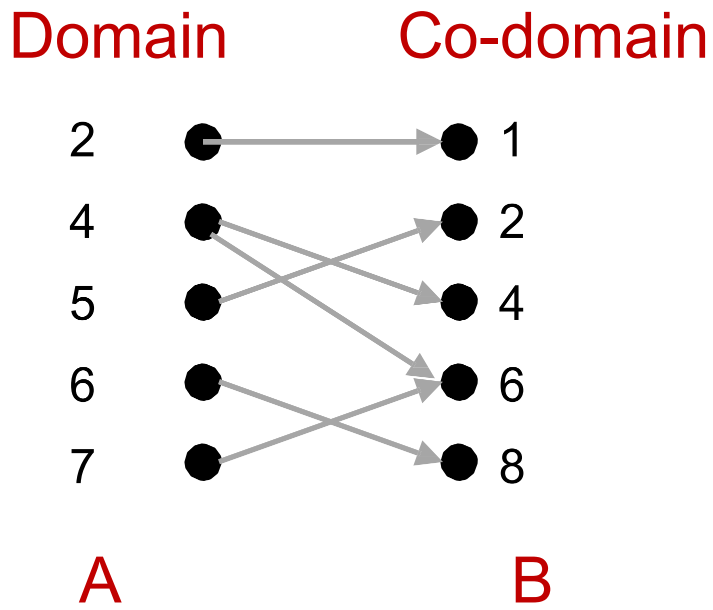

Not a valid function

# Functions and Non-Functions

- Which of the arrow diagrams define functions from
- A = {2,4,5,6,7} to B = {1,2,4,6,8}.




Not a Function




Not a Function

# Example

- Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$

  assign the square of an integer to this integer
- What is $f(x)$ =?
  - $f(x) = x^2$
- What is domain of $f$ ?
  - Set of all integers
- What is codomain of $f$ ?
  - Set of all integers
- What is the range of $f$ ?
  - $\{0, 1, 4, 9, \ldots\}$. All integers that are perfect squares

# Function Arithmetic

- Just as we are able to **add (+), subtract (-), multiply (*), and divide (÷)** two or more numbers, we are able to **+ , - , * , and ÷ two or more functions**

- Let *f* and *g* be functions from ***A*** to **R**. Then $f+g$, $f-g$, $f*g$ and $f/g$ are also functions from ***A*** to **R** defined for all $\boldsymbol{x} \in \boldsymbol{A}$ by:
- $(f+g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f*g)(x) = f(x)*g(x)$
- $(f/g)(x) = f(x)/g(x)$ given that $g(x) \neq 0$

# Function Arithmetic

Let $f_1$ and $f_2$ be functions from **R** to **R** such that:

- $f_1(x) = 2x$
- $f_2(x) = x^2$
- Find $f_1 + f_2$ and $f_1 * f_2$.
- $f_1 + f_2 = (f_1 + f_2)(x) = f_1(x) + f_2(x) = 2x + x^2$
- $f_1 * f_2 = (f_1 * f_2)(x) = f_1(x) * f_2(x) = 2x * x^2 = 2x^3$

# Function Arithmetic

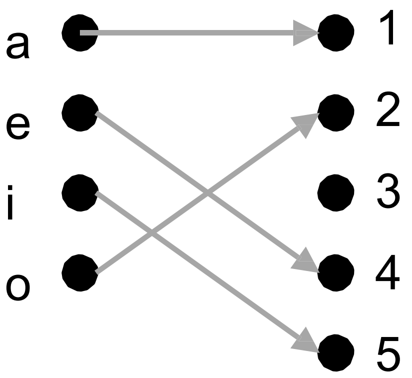
- Let f and g be functions from **R** to **R** such that:
- $\mathbf{f(x) = 3x+2}$ $\qquad\qquad$ $\mathbf{g(x) = -2x + 1}$
- What is the function $f*g$?
- $f*g = (f*g)(x) = f(x)*g(x) = (\mathbf{3x}+2)*(-2x+1) = -6x^2 - x + 2$

Let $x = -1$, what is $f(-1)*g(-1)$ and $(f*g)(-1)$?

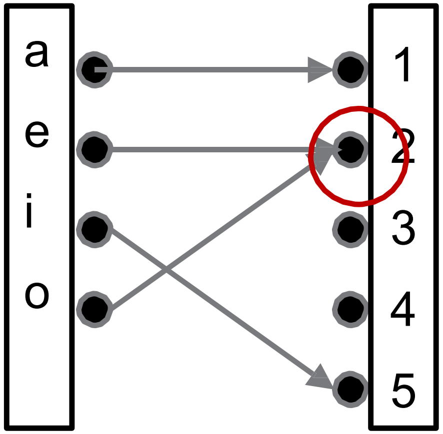
- $f(-1) = 3(-1) + 2 = -1$
- $g(-1) = -2(-1) + 1 = 3$
- $f(-1)*g(-1) = -1\times 3 = -3$
- $(f*g)(-1) = -6(-1)2 - (-1) + 2 = -6+1+2 = -3$

# One-to-One Function

- A function is one-to-one if each element in the co-domain has a unique pre-image
- Formal definition: A function ***f*** is one-to-one if f(a) = f(b) implies a = b for all a and b in the domain of ***f***.




A one-to-one function




A function that is not one-to-one

# One-to-One Function

$f$ is one-to-one using quantifiers/symbols

$\forall a, \forall b\ [\ f(a) = f(b) \rightarrow a = b\ ]$ or equivalently

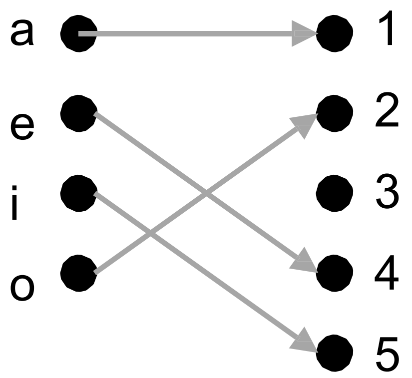
$$\forall a, \forall b, [a \neq b \rightarrow f(a) \neq f(b)]$$

Where $a$ and $b$ are in domain of $f$ i.e. $\forall a, b \in D(f)$

# More on One-to-One Functions

- Injective is synonymous with one-to-one
    - “A function is injective”
- A function is an injection if it is one-to-one

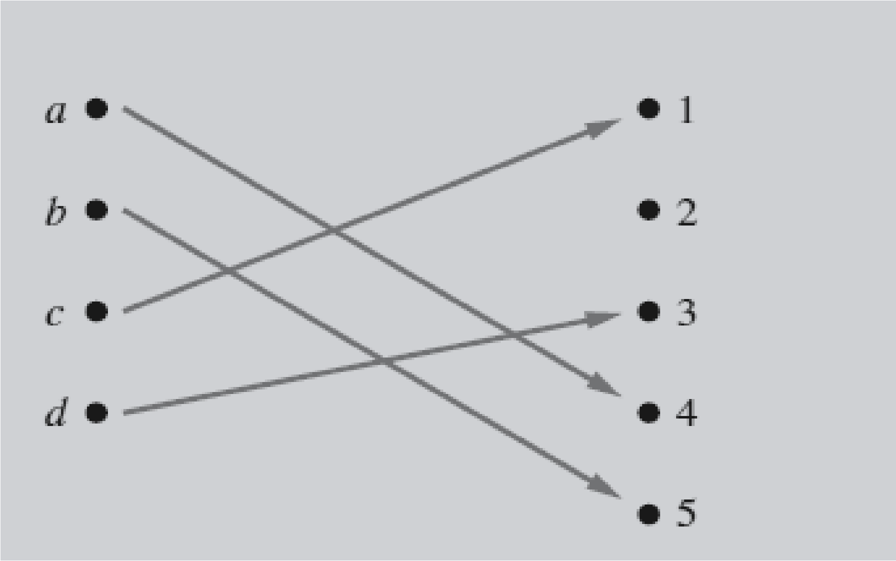
- Note that there can be un-used elements in the co-domain




A one-to-one function

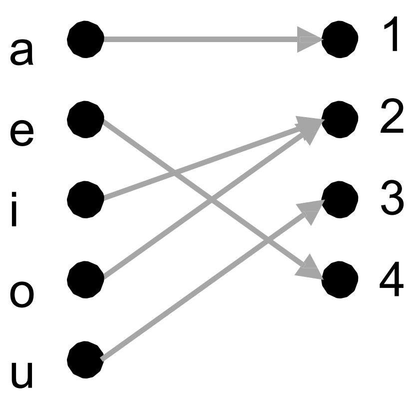
# Example one-to-one function

- Determine whether the function $f$ from set $\{a, b, c, d\}$ to set $\{1,2,3,4,5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1$ $and$ $f(d) = 3$ $is$ $one - to - one$?

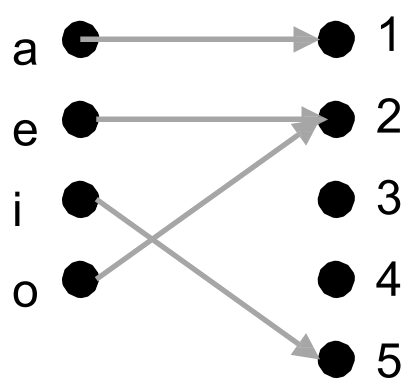


# Onto Functions

- A function is onto if each element in the co-domain is an image of some pre-image
- Formal definition: A function f is onto if for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.




An onto function




A function that is not onto

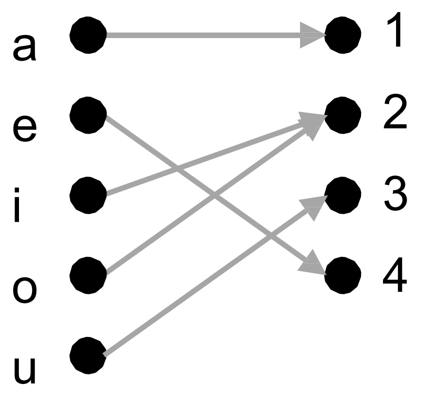
# Onto functions

- A function $f$ is onto if $\forall\ b, \exists\ a, [\ f(a) = b]$ , where the domain for $a$ is the domain of the function and the domain for $b$ is the codomain of the function.

# More on Onto Functions

- Surjective is synonymous with onto
  - “A function is surjective”
- A function is a surjection if it is onto

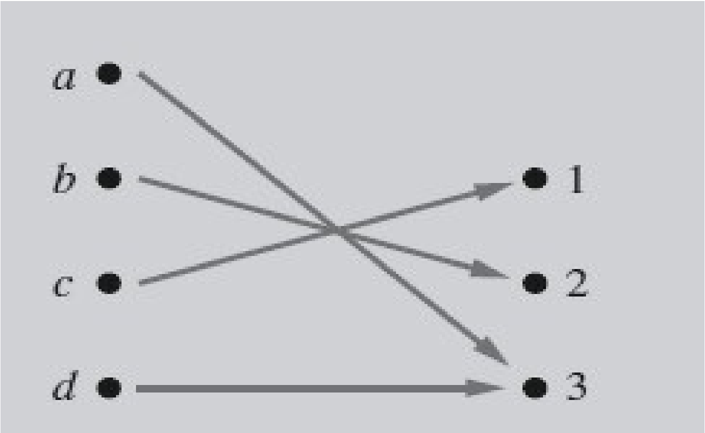
- Note that there can be multiple used elements in the co-domain




An onto function

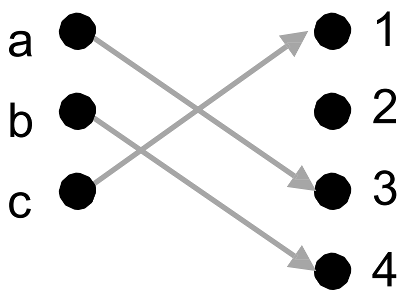
# Example onto function

- Determine whether the function $f$ from $\{a, b, c, d\}$ to $\{1,2,3\}$ $defined\ by\ f(a) = 3, f(b) = 2, f(c) = 1\ and\ f(d) = 3. Is\ f\ an\ onto\ function?$

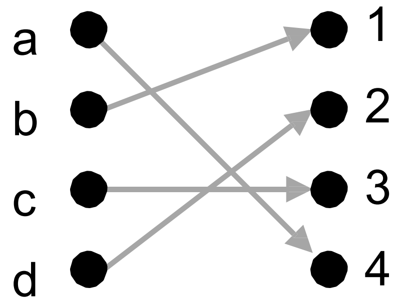


# Onto vs One-to-One

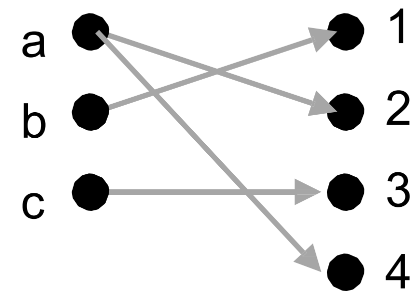
- Are the following functions onto, one-to-one, both, or neither?



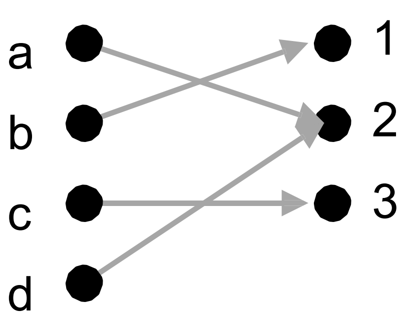

1-to-1, not onto



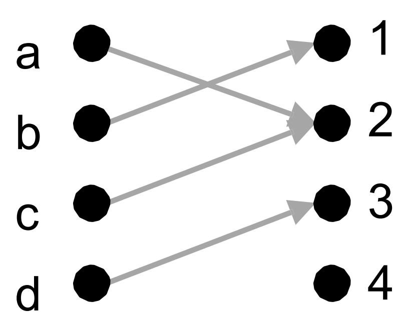

Both 1-to-1 and onto




Not a valid function




Onto, not 1-to-1




Neither 1-to-1 nor onto

# Example

- Determine whether the function *f (x)* = $x^2$ from the set of integers to the set of integers is one-to-one.

$$f:\mathbb{Z} \to \mathbb{Z};\ f(x) = x^2$$

To show $f$ is one to one, let $x_{1,} x_2 \in$ Z and suppose

$$f(x_1) = f(x_2)$$

$$x_1^2 = x_2^2$$

$$x_1^2 - x_2^2 = 0$$

$$(x_1 + x_2)(x_1 - x_2) = 0$$

$$x_1 + x_2 = 0, \quad x_1 - x_2 = 0$$

$$x_1 = -x_2, \qquad x_1 = x_2$$

$$x_{1=} \pm x_2$$

Hence $f$ is not one to one.

# Example

- Determine whether the function *f (x)* = $x + 1$ from the set of integers to the set of integers is onto.

$$f:\mathbb{Z} \to \mathbb{Z}; \; ; \; f(x) = x + 1$$

Let y $\in$ Z. We look for an x $\in$ Z such that f(x) = y

$f(x) = y$

$x + 1 = y$ By definition of $f$

$x = y - 1$

Thus for each $y \in$ Z, there exists $x = y - 1 \in$ Z

such that $f(x) = f(y - 1)$

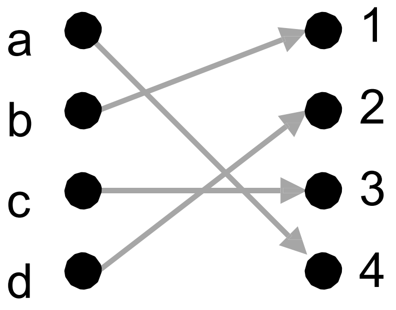
$= (y - 1) + 1 = y$

Hence f is onto.

# Bijections

- Consider a function that is both one-to-one and onto:

- Such a function is a one-to-one correspondence, or a bijection.



# Example

- Determine whether the following functions are bijective or not?

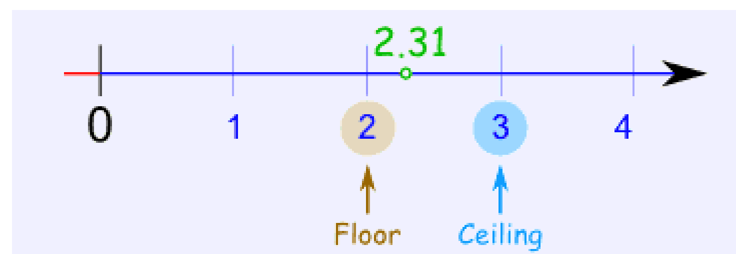
$$f: R \rightarrow R;\ f(x) = x^3$$

$$f: R - \{0\} \rightarrow R;\ f(x) = \frac{x+1}{x}$$

# Floor and Ceiling

- floor($x$) = $\lfloor x \rfloor$ is the largest integer that is less than or equal to **x** and ceiling($x$) = $\lceil x \rceil$ is the smallest integer that is greater than or equal to **x**
- The floor and ceiling functions give you the **nearest integer** up or down.

| **Sample value $x$** | **Floor $x$** | **Ceiling $x$** |
|---|---|---|
| 12/5 = 2.4 | 2 | 3 |
| 2.7 | 2 | 3 |
| −2.7 | −3 | −2 |
| −2 | −2 | −2 |



# Identity Functions

- A function such that the image and the pre-image are ALWAYS equal

- $f(x) = 1*x$
- $f(x) = x + 0$

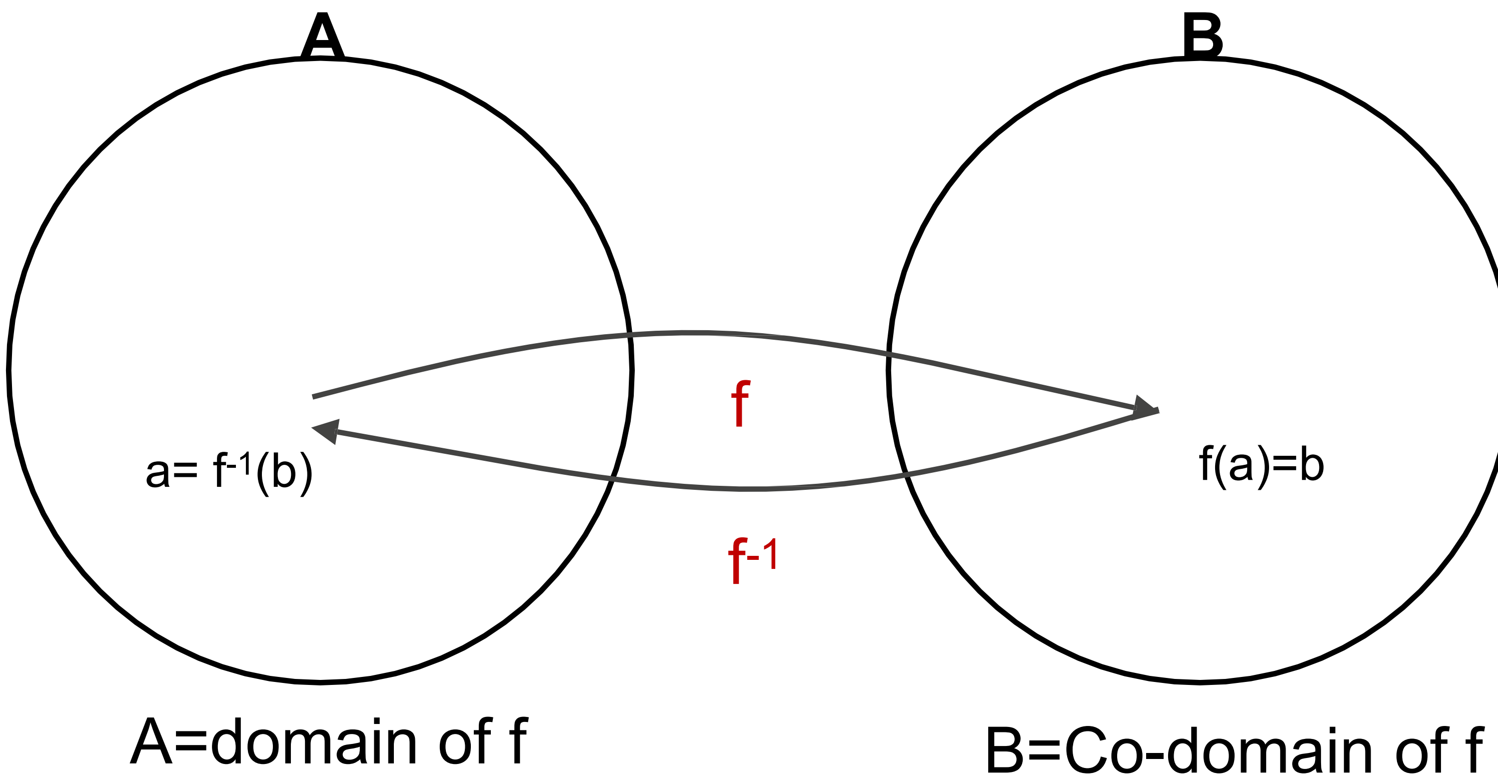
- The domain and the co-domain must be the same set.

# Inverse Function

- Let ***f*** be a one-to-one correspondence from the set **A** to the set **B**.
- The inverse function of ***f*** is the function that assigns to an element in **b** belonging to **B** the unique element **a** in **A** such that ***f*(a) = b**.
- The inverse function of ***f*** is denoted by $\boldsymbol{f^{-1}}$.
- Hence , $\boldsymbol{f^{-1}}$**(b) = a** when ***f*(a) = b**.

# Inverse Functions

If $f(a) = b$, then $f^{-1}(b) = a$

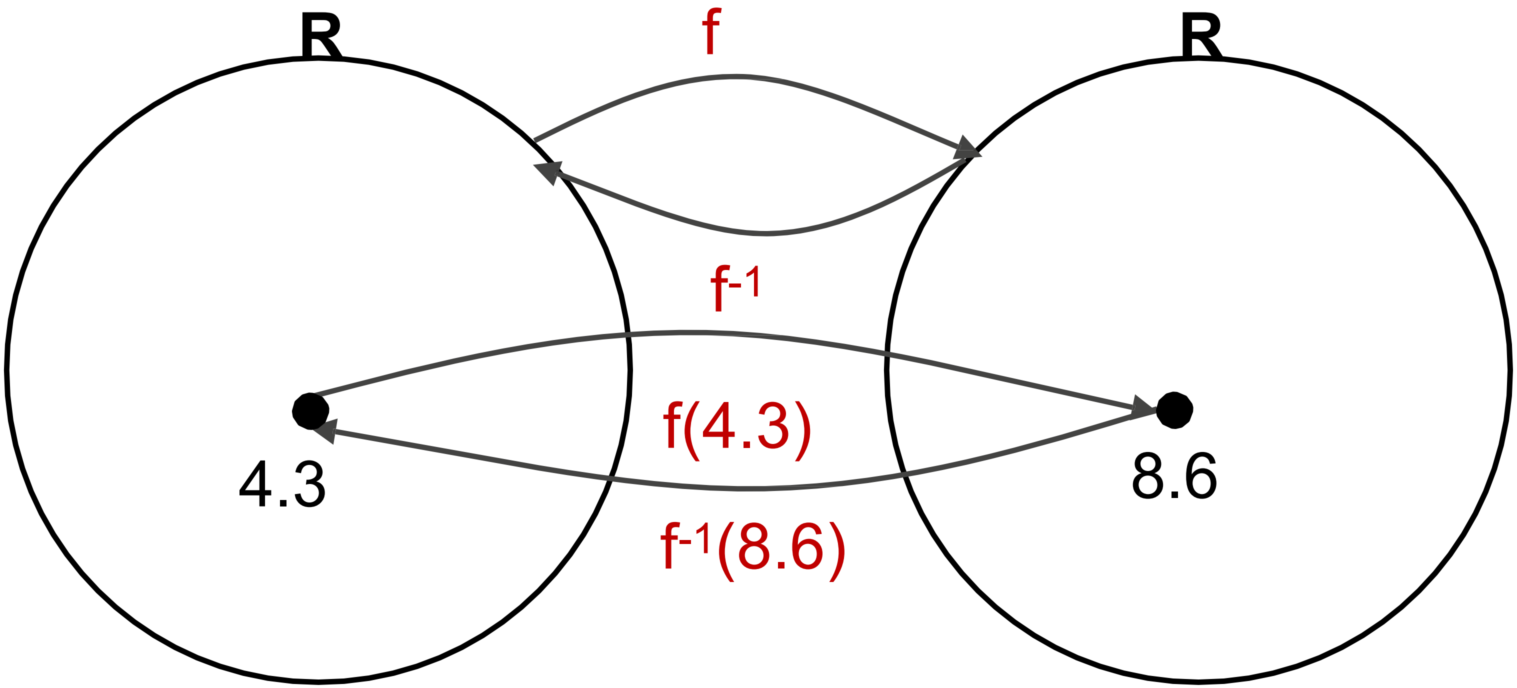


# Inverse Functions

If $f(x) = y$, then $f^{-1}(y) = x$

Let $f(x) = 2*x$

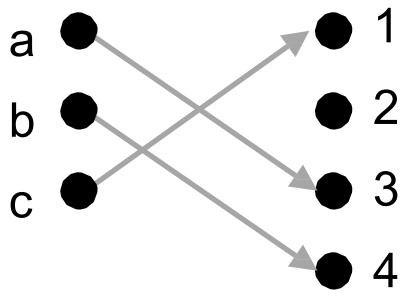
$f(x)=y$

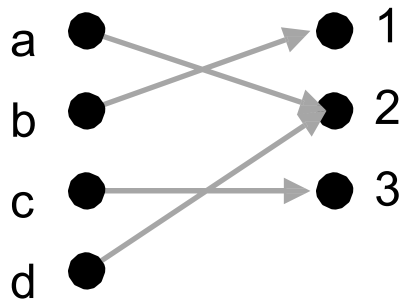



Then $f^{-1}(y) = y/2$

# More on Inverse Functions

- Can we define the inverse of the following functions?




What is $f^{-1}(2)$?
Not onto!

What is $f^{-1}(2)$?
Not 1-to-1!

- An inverse function can ONLY be done defined on a bijection

# More on Inverse Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

# Example

- Let $f$ be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.
- Is $f$ invertible, and if it is, what is its inverse?

# Working Rule to Find Inverse Function

- Let f: X →Y be a one-to-one correspondence defined by the formula f(x) = y.

1. Solve the equation f(x) = y for x in terms of y.
2. $\mathbf{f}^{-1}$ (y) equals the right hand side of the equation found in step 1.

# Example

Let f : Z → Z be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?

The given function $f$ is defined by rule

$$f(x) = y$$

$$x + 1 = y$$

$$x = y - 1$$

Hence $f^{-1}(y) = y - 1$

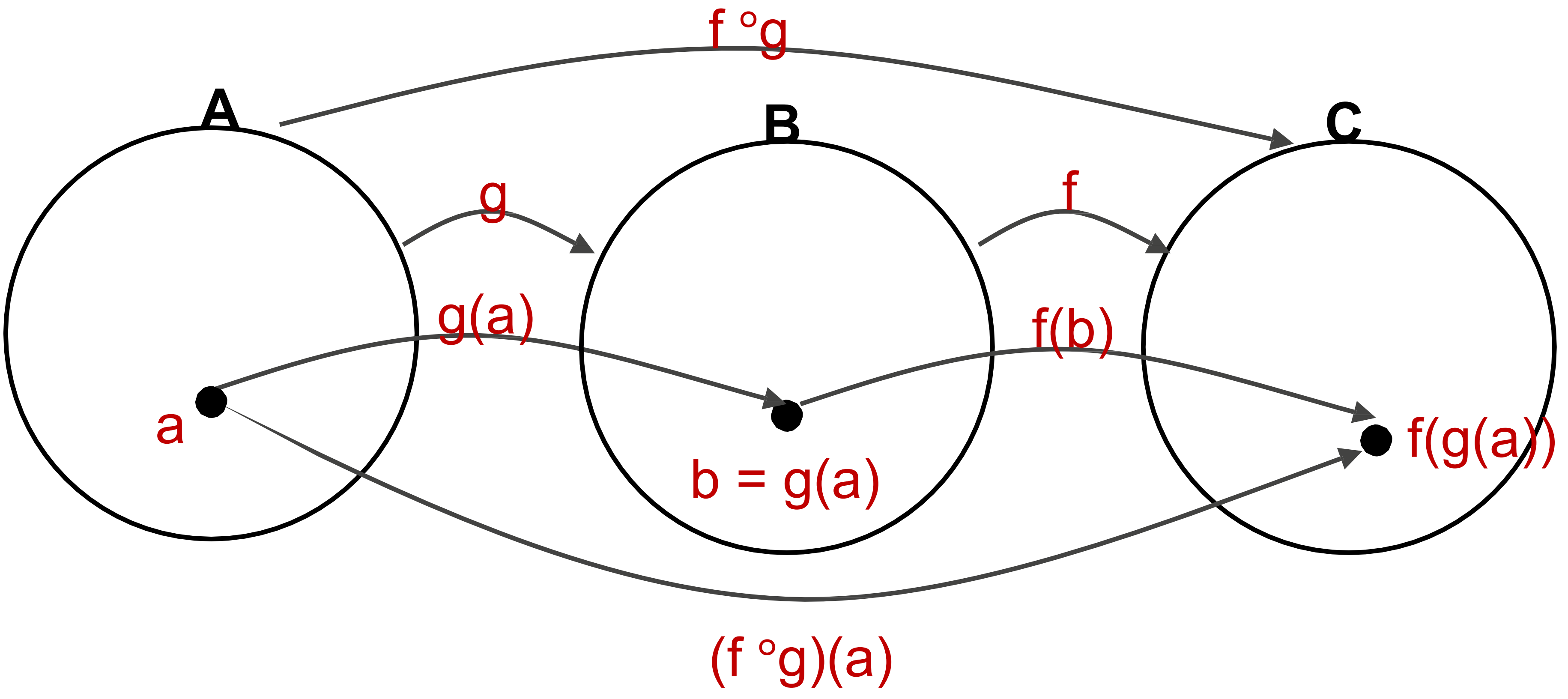
# Compositions of Functions

- Let ***g*** be a function from the set **A** to the set **B** and let ***f*** be a function from the set **B** to the set **C**. the compositions of the functions f and g, denoted by ***f*** ∘***g***, is defined by

$$(f \circ g)(a) = f\,(g(a))$$

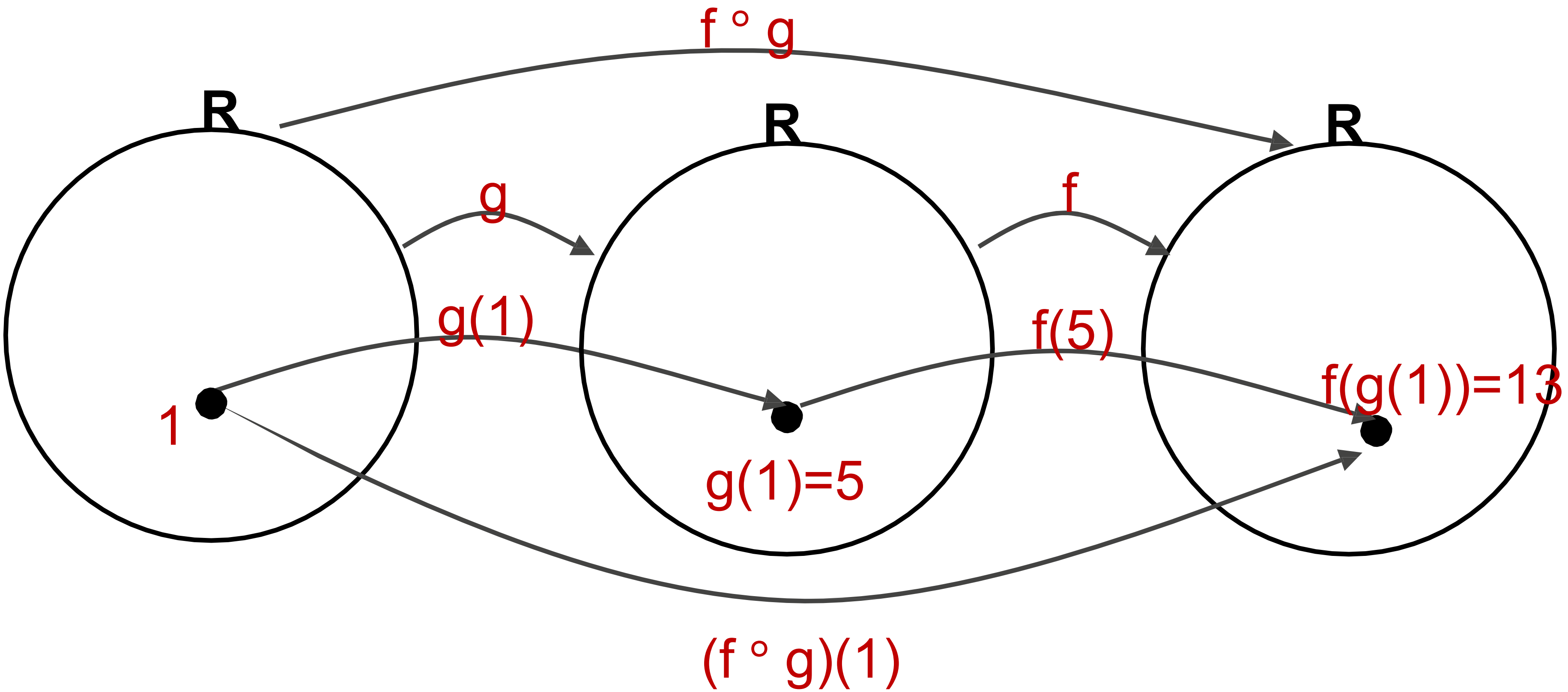
# Compositions of Functions

$$(f \circ g)(a) = f(g(a))$$



# Compositions of Functions

Let $f(x) = 2x+3$ Let $g(x) = 3x+2$




$f(g(x)) = 2(3x+2)+3 = 6x+7$

# Compositions of Functions

Does f(g(x)) = g(f(x))?

Let $f(x) = 2x+3$

Let $g(x) = 3x+2$

$f(g(x)) = 2(3x+2)+3 = 6x+7$

$g(f(x)) = 3(2x+3)+2 = 6x+11$

Not equal!

Function composition is not commutative!

# Compositions of Functions

- Let A = {1,2,3,4,5}

  $f : A \to A$ and $g : A \to A$

  $f(1) = 3,\ f(2) = 5,\ f(3) = 3,\ f(4) = 1,\ f(5) = 2$

  $g(1) = 4,\ g(2) = 1,\ g(3) = 1,\ g(4) = 2,\ g(5) = 3$

  Find the composition functions $f \circ g$ and $g \circ f$.

| $f \circ g$ | $g \circ f$ |
|---|---|
| $(f \circ g)(1) = f(g(1)) = f(4) = 1$ | $(g \circ f)(1) = g(f(1)) = g(3) = 1$ |
| $(f \circ g)(2) = ?$ | $(g \circ f)(2) = ?$ |
| $(f \circ g)(3) = ?$ | $(g \circ f)(3) = ?$ |
| $(f \circ g)(4) = ?$ | $(g \circ f)(4) = ?$ |
| $(f \circ g)(5) = ?$ | $(g \circ f)(5) = ?$ |

# Compositions of Functions

Let $g : A \rightarrow A$ be the function, Set A = {$a, b, c$} such that $g(a) = b$, $g(b) = c$, and $g(c) = a$.

Let $f : A \rightarrow B$ be the function, Set A = {$a, b, c$} to the set B = {1, 2, 3} such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of $f$ and $g$, and what is the composition of $g$ and $f$?

Solution:

The composition $f \circ g$ is defined by

$(f \circ g)(a) = f(g(a)) = f(b) = 2$,

$(f \circ g)(b) = f(g(b)) = f(c) = 1$,

$(f \circ g)(c) = f(g(c)) = f(a) = 3$.

$g \circ f$ is not defined, because the range of $f$ is not a subset of the domain of $g$.

# Exercise Questions

Chapter # 2

Topic # 2.3

Questions 1, 2, 8, 9,10,11,12, 22, 23, 36, 37