Discrete Structures
(Discrete Mathematics)

Lecture-4

Introduction
Propositional Logic

Presentation Outline

- Statement
- Propositional Logic and First Order Logic
- Formal Logic
- Quantifiers and Predicates

Text Books – A reminder

- Discrete Mathematics and its Applications 7th Ed. by Kenneth H. Rosen, McGraw Hill Publisher.

- Discrete Mathematics with Applications 4th Ed. by Susanna S., Thomson Learning, Inc.

Logic

Logic is the study of the principles and methods that distinguishes between a valid and an invalid argument.

Logic deals with general reasoning laws, which you can trust.

Applications

- Applied in proving **program correctness** and **verification**
- Databases (Relational Algebra and calculus)
- Artificial Intelligence
- Many more within Computer Science and Software Engineering

Propositional Logic

- Proposition
  - A proposition is a declarative statement that is either TRUE or FALSE, but not both.
- Example 1
  - 2 + 2 = 4.
  - Lahore is the capital of Pakistan.
  - It is Sunday today.
  - Ali is student of this class.
- Example 2
  - What time is it?
  - X + 1 = 2.
  - Close the door.
  - Read this carefully.

Propositional Logic

- Letter are used to denote propositional variables, to symbolically represent propositions.
  - Letters used for this purpose are p, q, r, s,………
  - A propositional can have one of two values: true (T) or false (F).

- Example
  - ***p* =** “Islamabad is the capital of Pakistan”
  - ***q* =** “17 is divisible by 3”

Propositional Variable

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- Example
  - ***p* =** "Islamabad is the capital of Pakistan"
  - ***q* =** "17 is divisible by 3"

Propositional Logic

- The area of logic that deals with propositions is called the *Propositional Calculus* or *Propositional Logic.*

- *Compound Propositions* are constructed by combining one or more propositions using logical operators (connectives).

- Examples
  - "3 + 2 = 5" **and** "Lahore is a city in Pakistan"
  - "The grass is green" **or** " It is hot today"

Compound Proposition

- *Compound Propositions* are constructed by combining one or more propositions using *logical operators* (connectives).

- Examples
  - “3 + 2 = 5” **and** “Lahore is a city in Pakistan”
  - “The grass is green” **or** “ It is hot today”

Symbols for Logical Operators

Symbol	Meaning
¬	Negation
∧	And, Conjunction
∨	Or, Disjunction
→	Implication
↔	Bi-Conditional

Logical Operators (Logical connectives)

- Negation
  - This just turns a false proposition to true and the opposite for a true proposition.
  - Symbol: ¬
  - Let *p* is a proposition. The statement

"It is not the case that p."

is another proposition, called the negation of p.

  - The negation of *p* is written ¬*p* and read as "not p".

Logical Operator - Negation

- Logical operators are defined by **truth tables** –tables which give the output of the operator in the right-most column.
- Here is the truth table for negation:

p	¬p
T	F
F	T

Logical Operator - Negation

- Example

Let p = “Today is Friday.”

The negation of p is

$\neg p$ = “It is not the case that today is Friday.”

$\neg p$ = “Today is not Friday.”

$\neg p$ = “It is not Friday today.”

- What is negation of following proposition: “My PC runs Linux”

Logical Operator - Conjunction

- Conjunction is a *binary* operator in that it operates on two propositions when creating compound proposition. On the other hand, negation is a *unary* operator.
- Conjunction corresponds to English “and.”
- Symbol: $\land$
- Let p and q be propositions. The conjunction of p and q, denoted by $p \land q$, is the proposition “p and q”. The conjunction $p \land q$is true when both p and q are true. If one of these is false, than the compound statement is false as well.

Logical Operator - Conjunction

- Truth Table

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operator - Conjunction

- Example

Let p = “Today is Friday.”

and q = “It is raining today.”

$p \wedge q$ = “Today is Friday and it is raining today.”

Logical Operator - Conjunction

- Hamza’s PC has more than 16 GB free hard disk space, and the processor in Hamza’s PC runs faster than 1 GHz.

- It is cold but sunny.

Logical Operator - Disjunction

- Disjunction is also a binary operator.
- Disjunction corresponds to English "or."
- Symbol: $\vee$

- Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q". The conjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Logical Operator - Disjunction

- Truth Table

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operator - Disjunction

- Example

Let p = “Today is Friday.”

and q = “It is raining today.”

$p \vee q$ = “Today is Friday or it is raining today.”

Example

Let p = "it is hot",

q = "it is sunny"

- It is hot and sunny $p \wedge q$
- It is not hot but sunny $\neg p \wedge q$
- It is neither hot nor sunny $\neg p \wedge \neg q$

Logical Operator – Exclusive Or

- Symbol: $\oplus$
- Let p and q be propositions. The exclusive or of p and q, denoted by p $\oplus$ q , is the proposition that is true when exactly one of p and q is true and is false, and false otherwise.
- Truth Table

p	q	p$\oplus$ q
T	T	F
T	F	T
F	T	T
F	F	F

Logical Operator – Exclusive Or

- Example

Let p = “Students who have taken calculus can take this class.”

and q = “Students who have taken computer science can take this class.”

p ∨ q = “Students who have taken calculus or computer science can take this class.”

p ⊕ q = “Students who have taken calculus or computer science, but not both, can enroll in this class.”

Exclusive or Versus Inclusive or (Disjunction)

- Coffee or tea comes with dinner. Exclusive or
- A password must have at least three digits or be at least five characters long. Inclusive or
- Lunch includes soup or salad. Exclusive or
- Experience with C++ or Java is required. Inclusive or

Logical Operator – Implication

- $p \rightarrow q$ corresponds to English "if p then q," or "p implies q."
- Symbol: $\rightarrow$
- The implication p $\rightarrow$ q is the proposition that is false when p is true and q is false, and true otherwise.

$$p \rightarrow q$$

Hypothesis Conclusion

- Examples
  - If it is raining then it is cloudy.
  - If you get 100% on the final, then you will get an A.
  - If p then 2+2 = 4.

Logical Operator – Implication

- Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Operator – Implication

- Alternate ways of stating an implication
  - *p* implies *q*
  - If *p*, *q*
  - *p* only if *q*
  - *p* is sufficient for *q*
  - *q* if *p*
  - *q* whenever *p*
  - *q* is necessary for *p*

Implication - Example

p: you get 100% on the final
q: you will get an A

- **p implies that q.**
  you get 100% on the final implies that you will get an A.
- **If p, then q.**
  If you get 100% on the final, then that you will get an A.
- **If p, q.**
  If you get 100% on the final, that you will get an A.
- **p is sufficient for q.**
  Get 100% on the final is sufficient for getting an A.
- **q if p.**
  you will get an A if you get 100% on the final.
- **q unless ¬ p.**
  you will get an A unless you don't get 100% on final.

Logical Operator – Implication

- Converse

  The proposition $q \rightarrow p$ is **converse** of $p \rightarrow q$.

- Contrapositive

  The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

- Inverse

  The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$.

Logical Operator – Implication

- Example

  “The home team wins whenever it is raining?”

Because “*q* whenever *p*”, so $p \rightarrow q$, the original statement can be rewritten as “If it is raining, then the home team wins.”

- Contrapositive

  “If the home team does not win, then it is not raining.”

- Converse

  “If the home team wins, then it is raining.”

- Inverse

  “If it is not raining, then the home team does not win.”

Logical Operator – Bi-conditional

- $p \leftrightarrow q$ corresponds to English "p if and only if q."
- Symbol: $\leftrightarrow$
- The bi-conditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
- Bi-conditional statements are also called *bi-implications*.
- Alternatively, it means "(if p then q) and (if q then p)"

- Example
  - "You can take the flight if and only if you buy a ticket."

Logical Operator – Bi-conditional

- Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Operator – Bi-conditional

p: You can take flight

q: You buy a ticket

$p \leftrightarrow q$

You can take flight if and only if you buy a ticket

What is the truth value when:

- you buy a ticket and you can take the flight ??
- $T \leftrightarrow T \equiv T$
- you don't buy a ticket and you can't take the flight ??
- $F \leftrightarrow F \equiv T$
- you buy a ticket but you can't take the flight ??
- $T \leftrightarrow F \equiv F$
- you can't buy a ticket but can take the flight ??
- $F \leftrightarrow T \equiv F$

Logical Operator – Bi-conditional

- Other English equivalents:
  - “p if and only if q”
  - “p is equivalent to q”
  - “p is necessary and sufficient for q”
  - “p iff q”
  - “If p then q, and conversely”

Bi-conditional -Example

p: "You can take the flight"

q: "You buy a ticket"

$p \leftrightarrow q$:

You can take the flight if and only if you buy a ticket

You can take the flight iff you buy a ticket

The fact that you can take the flight is necessary and sufficient for buying a ticket

Logical Operators Summary

		Not	Not	And	Or	Xor	Implication	Bi-conditional
p	**q**	**$\neg p$**	**$\neg q$**	**$p \wedge q$**	**$p \vee q$**	**$p \oplus q$**	**$p \rightarrow q$**	**$p \leftrightarrow q$**
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

Truth Table of Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the every propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
  - This includes the atomic propositions

Truth Table of Compound Propositions

- $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Truth Table of Compound Propositions

- $p \rightarrow (\neg q \wedge r)$

p	q	r	$\neg q$	$\neg q \wedge r$	$p \rightarrow (\neg q \wedge r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	F	T

Precedence of Logical Operators

- Just as in algebra, operators have precedence

$$4+3*2 = 4+(3*2), \qquad \text{not } (4+3)*2$$

- Example

This means that

$p \vee q \wedge \neg r \rightarrow s \leftrightarrow t$

yields: $(p \vee (q \wedge (\neg r)) \rightarrow s) \leftrightarrow (t)$

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Truth Tables

- Construct the truth table of following compound propositions
  - $p \rightarrow \neg p$
  - $p \oplus p$
  - $(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$

Chapter Reading

- ***Chapter 1***, Kenneth H. Rosen, Discrete Mathematics and Its Applications

Chapter Supplementary Exercise (For Practice)

- Question # 1, 2, 3, 4, 8, 9, 13, 24, 27, 28, 31, 32