Linear Algebra

Batch 17-BS(Mathematics)

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Solution of Linear System using Elementary Row Operations

Matrices: overview

Matrices: Overview

• A *matrix* is simply a rectangular array of numbers.

 Matrices are used to organize information into categories that correspond to the rows and columns of the matrix.

Matrices: Overview

 For example, a scientist might organize information on a population of endangered whales as follows:

	Immature	Juvenile	Adult			
Male	$\begin{bmatrix} 12\\15 \end{bmatrix}$	52	18			
Female		42	11			

 This is a compact way of saying there are 12 immature males, 15 immature females, 18 adult males, and so on. Matrices and Systems of Linear Equations

Introduction

• A Linear system is represented by a matrix.



- This matrix is called the <u>augmented matrix</u> of the linear system.
- The augmented matrix contains the same information as the system, but in a simpler form.
- The operations we learned for solving systems of equations can now be performed on the augmented matrix.

More on Matrices

Matrices

• We begin by defining the various elements that make up a matrix.

Matrix—Definition

 An m x n matrix is a rectangular array of numbers with m rows and n columns.



Matrix—Definition

• We say the matrix has dimension *m* x *n*.

 The numbers *a_{ij}* are the entries of the matrix.

The subscript on the entry *a_{ij}* indicates that it is in the *i*th row and the *j*th column.

Examples

• Here are some examples.

Matrix	Dimension	
1 3 0 2 4 -1	2 x 3	2 rows by 3 columns
[6 –5 0 1]	1 x 4	1 row by 4 columns

The Augmented Matrix of a Linear System

Augmented Matrix

- We can write a system of linear equations as a matrix by writing only the coefficients and constants that appear in the equations.
 - This is called the augmented matrix of the system.

Augmented Matrix

• Here is an example.

Linear System	Augmented Matrix							
$\int 3x - 2y + z = 5$	□ 3 −2 1 5							
$\begin{cases} x+3y-z=0 \end{cases}$	1 3 -1 0							
$\begin{vmatrix} -x+ & 4z=11 \end{vmatrix}$	_1 0 4 11							

 Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

Finding Augmented Matrix of Linear System

• Write the augmented matrix of the system of equations.

$$\begin{cases} 6x - 2y - z = 4\\ x + 3z = 1\\ 7y + z = 5 \end{cases}$$

Finding Augmented Matrix of Linear System

• First, we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 0 \\ 7y + z = 5 \end{cases}$$

Finding Augmented Matrix of Linear System

 The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

Elementary Row Operations

Elementary Row Operations

- 1. Interchange two rows.
- 2. Multiply a row by a nonzero constant.
- 3. Add a multiple of one row to another.
- Note that performing any of these operations on the augmented matrix of a system does not change its solution.

Elementary Row Operations preserve linear system in question

- 1. Swapping rows is just changing the order of the equation, which certainly should not change solution.
- Scalar multiplication of a row is just multiplying the equation by the same number on both sides.

Note that each row corresponds to one equation of linear system.

Elementary Row Operations preserve linear system in question

3. Adding one row to other does not change solution, as both equations (corresponding to rows) share the solution.

Elementary Row Operations—Notation

We use the following notation to describe the elementary row operations:

Symbol	Description
$R_i \leftrightarrow R_j$	Interchange the <i>i</i> th and <i>j</i> th rows.
<i>k</i> R _i	Multiply the <i>i</i> th row by <i>k</i> .
$R_i + kR_j \rightarrow R_i$	Change the <i>i</i> th row by adding <i>k</i> times row <i>j</i> to it. Then, put the result back in row <i>i</i> .

Elementary Row Operations

 In the next example, we compare the two ways of writing systems of linear equations.

Elementary Row Operations and Linear System

• Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

 Our goal is to eliminate the *x*-term from the second equation and the *x*- and *y*-terms from the third equation.

Elementary Row Operations and Linear System For comparison, we write both the system of •equations and its augmented matrix.

System	Augmented Matrix							
$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$							
$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_{3}-3R_{1}\rightarrow R_{3}]{1 -1 3 4} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$							

Elementary Row Operations and Linear System

Elementary Row Operations and Linear System

 Now, we use back-substitution to find that:

$$x = 2, y = 7, z = 3$$

- The solution is (2, 7, 3).

Gaussian Elimination

Gaussian Elimination

 In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations to arrive at a matrix in a certain form.

This form is described as follows.

Row-Echelon Form

• A matrix is in row-echelon form if it satisfies the following conditions.

- The first nonzero number in each row (reading from left to right) is 1. This is called the leading entry.
- 2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
- 3. All rows consisting entirely of zeros are at the bottom of the matrix.

Reduced Row-Echelon Form

 A matrix is in reduced row-echelon form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

Row-Echelon & Reduced Row-Echelon Forms

• In the following matrices,

- The first is not in row-echelon form.
- The second is in row-echelon form.
- The third is in reduced row-echelon form.

– The entries in red are the leading entries.

Not in Row-Echelon Form

Not in row-echelon form:



Row-Echelon & Reduced Row-Echelon Forms: Examples

•	Row-Echelon Form					Reduced Row-Echelon Form						
	∏ 1	3	-6	10	0]		[1	3	0	0	0]	
	0	0	1	4	-3		0	0	1	0	-3	
	0	0	0	1	$\frac{1}{2}$		0	0	0	1	$\frac{1}{2}$	
	0 ↑	0	0 ↑	0 ↑	0_		0 _↑	0	0 ↑	0 ↑	0	
	Leading 1's shift to the right in successive rows.					Lea 0's belo	idinę abo ow t	g 1's ve a hem	hav nd	ve		

Putting in Row-Echelon Form

- We now discuss a systematic way to put a matrix in row-echelon form using elementary row operations.
 - We see how the process might work for a 3 x 4 matrix.
Putting in Echelon Form: Step 1

• Start by obtaining 1 in the top left corner.

 Then, obtain zeros below that 1 by adding appropriate multiples of the first row to the rows below it.



Putting in Echelon Form: Steps 2 & 3

- Next, obtain a leading 1 in the next row.
- Then, obtain zeros below that 1.
 - At each stage, make sure every leading entry is to the right of the leading entry in the row above it.
 - Rearrange the rows if necessary.



Putting in Echelon Form:Step 4

• Continue this process until you arrive at a matrix in row-echelon form.



Gaussian Elimination

 Once an augmented matrix is in row-echelon form, we can solve the corresponding linear system using back-substitution.

 This technique is called Gaussian elimination, in honor of its inventor, the German mathematician C. F. Gauss.

Solving a System Using Gaussian Elimination

 To solve a system using Gaussian elimination, we use:

1. Augmented matrix

2. Row-echelon form

3. Back-substitution

Solving a System Using Gaussian Elimination

1. Augmented matrix

- Write the augmented matrix of the system.
- 2. Row-echelon form
 - Use elementary row operations to change the augmented matrix to row-echelon form.

Solving a System Using Gaussian Elimination

3. Back-substitution

 Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

Solving a System Using Row-Echelon Form

 Solve the system of linear equations using Gaussian elimination.

 $\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$

- We first write the augmented matrix of the system.
- Then, we use elementary row operations to put it in row-echelon form.

Solving a System Using Row-Echelon Form





Solving a System Using Row-Echelon Form $R_2 \rightarrow R_2 \rightarrow R_2$ $R_3 + 2R_1 \rightarrow R_3$ $R_3 + 2R_1 \rightarrow R_3$



Solving a System Using Row-Echelon Form $\xrightarrow{R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix}$ $\xrightarrow{-\frac{1}{10}R_3} \left[\begin{array}{cccc} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{array} \right]$

Solving a System Using Row-Echelon Form

- We now have an equivalent matrix in row-echelon form.
- The corresponding system of equations is:

 $\begin{cases} x+2y-z=1\\ y+4z=-7\\ z=-2 \end{cases}$

We use back-substitution to solve the system.

Solving a System Using Row-Echelon Form

•
$$y + 4(-2) = -7$$

• $y = 1$
• $x + 2(1) - (-2) = 1$
• $x = -3$

– The solution of the system is:

Gauss-Jordan Elimination

Putting in Reduced Row-Echelon Form

- If we put the augmented matrix of a linear system in reduced row-echelon form, then we don't need to back-substitute to solve the system.
 - To put a matrix in reduced row-echelon form, we use the following steps.
 - We see how the process might work for a 3 x 4 matrix.

Putting in Reduced Row-Echelon Form—Step 1

• Use the elementary row operations to put the matrix in row-echelon form.



Putting in Reduced Row-Echelon Form—Step 2

 Obtain zeros above each leading entry by adding multiples of the row containing that entry to the rows above it.

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \end{bmatrix}$$

Putting in Reduced Row-Echelon Form—Step 2

Begin with the last leading entry and work up.

$$\begin{bmatrix} 1 & - & - \\ 0 & 1 & - \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & - & 0 & - \\ 0 & 1 & 0 & - \\ 0 & 0 & 1 & - \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ 0 & 0 & 1 & - \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ 0 & 0 & 1 & - \end{bmatrix}$$

Gauss-Jordan Elimination

 Using the reduced row-echelon form to solve a system is called Gauss-Jordan elimination.

– We illustrate this process in the next example.

• Solve the system of linear equations, using Gauss-Jordan elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

 In Example 3, we used Gaussian elimination on the augmented matrix of this system to arrive at an equivalent matrix in row-echelon form.

 We continue using elementary row operations on the last matrix in Example 3 to arrive at an equivalent matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_2 - 4R_3 \to R_2]{R_1 + R_3 \to R_1} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$R_1 - 2R_2 \rightarrow R_1$$

- We now have an equivalent matrix in reduced row-echelon form.
- The corresponding system of equations is:

$$\begin{cases} x = -3 \\ y = 1 \\ z = -2 \end{cases}$$

− Hence, we immediately arrive at the solution (−3, 1, −2).

Inconsistent and Dependent Systems

Solutions of a Linear System

 The systems of linear equations that we considered in above examples had exactly one solution.

However, a linear system may have:
 One solution
 No solution
 Infinitely many solutions



Solutions of a Linear System

 Fortunately, the row-echelon form of a system allows us to determine which of these cases applies.

- First, we need some *terminology*.

Leading Variable

 A leading variable in a linear system is one that:

 Corresponds to a leading entry in the row-echelon form of the augmented matrix of the system.

Solutions of Linear System in Row-Echelon Form

 Suppose the augmented matrix of a system of linear equations has been transformed by Gaussian elimination into row-echelon form.

- Then, exactly one of the following is true.
 - No solution
 - One solution
 - Infinitely many solutions

Solutions of Linear System in Row-Echelon Form No solution:

- If the row-echelon form contains a row that represents
 the equation 0 = c where c
 is not zero, the system has
 no solution.
 - A system with no solution is called inconsistent.



Solutions of Linear System in Row-Echelon Form

- One solution:
 - If each variable in the row-echelon form is a leading variable, the system has exactly one solution.
 - We find this by using back-substitution or Gauss-Jordan elimination.



Solutions of Linear System in Row-Echelon Form • Infinitely many solutions:

- If the variables in the row-echelon form are not all leading variables, and if the system is not inconsistent, it has infinitely many solutions.
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- The system is called dependent.



Solutions of Linear System in Row-Echelon Form

 We solve the system by putting the matrix in reduced row-echelon form and then expressing the leading variables in terms of the non-leading variables.

 The non-leading variables may take on any real numbers as their values. System with No Solution (Example) Solve the system

$$\begin{cases} x-3y+2z=12\\ 2x-5y+5z=14\\ x-2y+3z=20 \end{cases}$$

- We transform the system into row-echelon form.

System with No Solution



The last matrix is in row-echelon form.

- So, we can stop the Gaussian elimination process.

System with No Solution $\begin{bmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Now, if we translate this last row back into equation form, we get 0x + 0y + 0z = 1, or 0 = 1, which is false.
- No matter what values we pick for x, y, and z, the last equation will never be a true statement.
- This means the system has no solution.
System with Infinitely Many Solutions (Example)

• Find the complete solution of the system.

$$\begin{cases} -3x - 5y + 36z = 10 \\ -x + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

We transform the system into reduced row-echelon form.

System with Infinitely Many Solutions (Example)



System with Infinitely Many Solutions (Example) $\begin{bmatrix} 1 & 1 & -7 & -5 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- The third row corresponds to the equation 0 = 0.
- This equation is always true, no matter what values are used for x, y, and z.
- Since the equation adds no new information about the variables, we can drop it from the system.

System with Infinitely Many Solutions (Examples)

- So, the last matrix corresponds to the system $\begin{cases} x & -7z = -5 \\ y - 3z = 1 \end{cases}$
 - Now, we solve for the leading variables
 x and y in terms of the non-leading
 variable z:

$$x = 7z - 5$$

 $y = 3z + 1$

System with Infinitely Many Solutions (Example)

 To obtain the complete solution, we let t represent any real number, and we express x, y, and z in terms of t:

•	<i>x</i> = 7 <i>t</i> – 5
•	y = 3t + 1
•	z = t

 We can also write the solution as the ordered triple (7t – 5, 3t + 1, t), where t is any real number.

• In Example, to get specific solutions we give a specific value to *t*.

$$x = 7(1) - 5 = 2$$

y = 3(1) + 1 = 4
z = 1

 Here are some other solutions of the system obtained by substituting other values for the parameter *t*.

Parameter <i>t</i>	Solution $(7t - 5, 3t + 1, t)$
$ \begin{array}{c} -1 \\ 0 \\ 2 \end{array} $	(-12, -2, -1) (-5, 1, 0) (9, 7, 2)
5	(30, 16, 5)

• Find the complete solution of the system.

 $\begin{cases} x + 2y - 3z - 4w = 10 \\ x + 3y - 3z - 4w = 15 \\ 2x + 2y - 6z - 8w = 10 \end{cases}$

We transform the system into reduced row-echelon form.



Since the last row represents
 the equation 0 = 0, we may discard it.

- So, the last matrix corresponds to the system $\begin{cases} x & -3z - 4w = 0 \\ y & = 5 \end{cases}$
 - To obtain the complete solution,
 we solve for the leading variables x and y
 in terms of the nonleading variables z and w,
 and we let z and w be any real numbers.

• Thus, the complete solution is:

•
$$x = 3s + 4t$$

- w = t
- where *s* and *t* are any real numbers.

 We can also express the answer as the ordered quadruple (3s + 4t, 5, s, t).

 Note that s and t do not have to be the same real number in the solution for Example.

 We can choose arbitrary values for each if we wish to construct a specific solution to the system.

For example, if we let s = 1 and t = 2, we get the solution (11, 5, 1, 2).

 You should check that this does indeed satisfy all three of the original equations in Example 7.

- Examples above illustrate this general fact:
 - If a system in row-echelon form has
 n nonzero equations in m variables (m > n), then the complete solution will have m n nonleading variables.

 For instance, in on of the above Examples, we arrived at two nonzero equations in the three variables x, y, and z.

• These gave us 3 - 2 = 1 nonleading variable.

Modeling with Linear Systems

Modeling with Linear Systems

 Linear equations—often containing hundreds or even thousands of variables—occur frequently in the applications of algebra to the sciences and to other fields.

 For now, let's consider an example that involves only three variables.

A Traffic Flow Problem

Modeling a traffic flow problem

Modeling a round-about



Can we find x_1 , x_2 , x_3 , x_4 and x_5 ?



We need to solve ...

$$x_1 - x_2 = 150 - 100 = 50$$

$$x_2 - x_3 = 50 - 150 = -100$$

$$x_3 - x_4 = 100 - 50 = 50$$

$$x_4 - x_5 = 300 - 400 = -100$$

$$x_5 - x_1 = 200 - 100 = 100$$

Representation using augmented matrix

Last time: Gaussian elimination

 Step 1: Try to transform the matrix into upper triangular form [* * * * *]

• Step 2: Backward substitution

• A nutritionist is performing an experiment on student volunteers.

- He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods:
 - MiniCal LiquiFast SlimQuick

- For the experiment, it's important that, every day, the subject consume exactly:
 - 500 mg of potassium
 - 75 g of protein
 - 1150 units of vitamin D

• The amounts of these nutrients in one ounce of each food are given here.

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

 How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

 Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day.

- This means that he will get:
 - 50x mg of potassium from MiniCal
 - 75y mg from LiquiFast
 - 10z mg from SlimQuick

• This totals 50x + 75y + 10z mg potassium.

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

• Based on the requirements of the three nutrients, we get the system

$$\begin{cases} 50x + 75y + 10z = 500 & Potassium \\ 5x + 10y + 3z = 75 & Protein \\ 90x + 100y + 50z = 1150 & Vitamin D \end{cases}$$

- Dividing the first equation by 5 and the third by 10 gives the system
 - $\begin{cases} 10x + 15y + 2z = 100\\ 5x + 10y + 3z = 75\\ 9x + 10y + 5z = 115 \end{cases}$
 - We can solve this using Gaussian elimination.
 - Alternatively, we could use a graphing calculator to find the reduced row-echelon form of the augmented matrix of the system.

- Using the rref command on the TI-83, we get the output shown.
 - From the reduced row-echelon form, we see that:

$$x = 5, y = 2, z = 10$$

rref(EA EE1 EO EO	0 1 0	0 5	5] 2] 1 0]]	
Figure 4					

- Every day, the subject should be fed:
 - 5 oz of MiniCal
 - 2 oz of LiquiFast
 - 10 oz of SlimQuick

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

Nutritional Analysis Using System of Linear Equations A more practical application might involve dozens of foods and nutrients rather than just three.

- Such problems lead to systems with large numbers of variables and equations.
- Computers or graphing calculators are essential for solving such large systems.