# Spline Interpolation Method

Numerical Analysis-I

BS (Mathematics), QUEST

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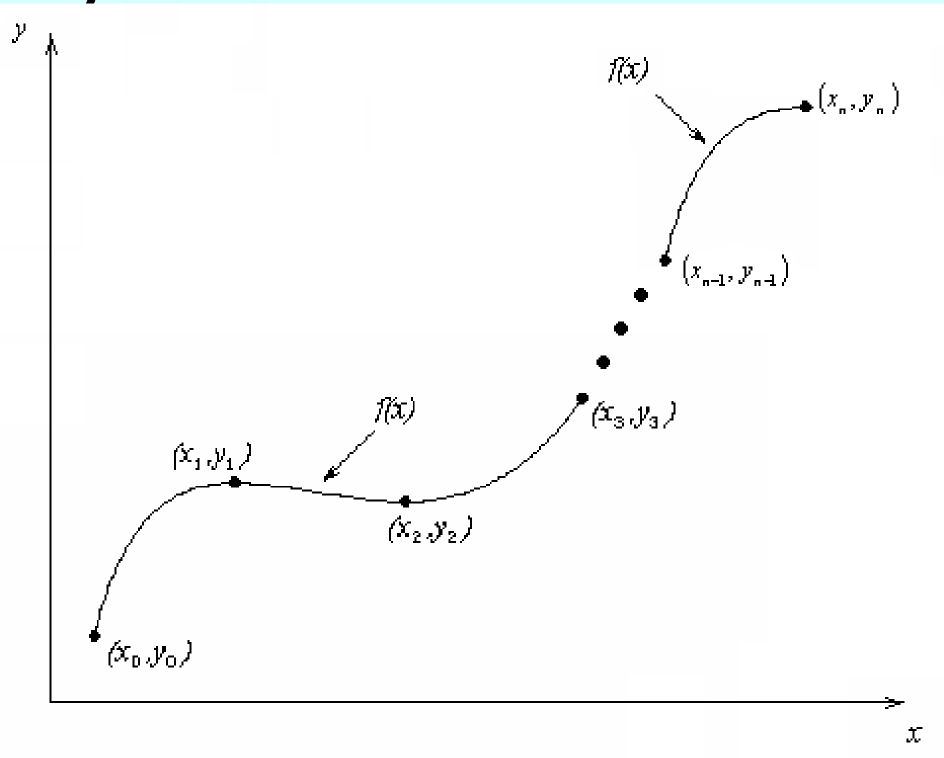
# What is Interpolation ?

Given $(x_0,y_0)$, $(x_1,y_1)$, …… $(x_n,y_n)$, find the value of 'y' at a value of 'x' that is not given.



# Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

# Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

**Table : Six equidistantly spaced points in [-1, 1]**

| $x$ | $y = \frac{1}{1+25x^2}$ |
|---|---|
| -1.0 | 0.038461 |
| -0.6 | 0.1 |
| -0.2 | 0.5 |
| 0.2 | 0.5 |
| 0.6 | 0.1 |
| 1.0 | 0.038461 |

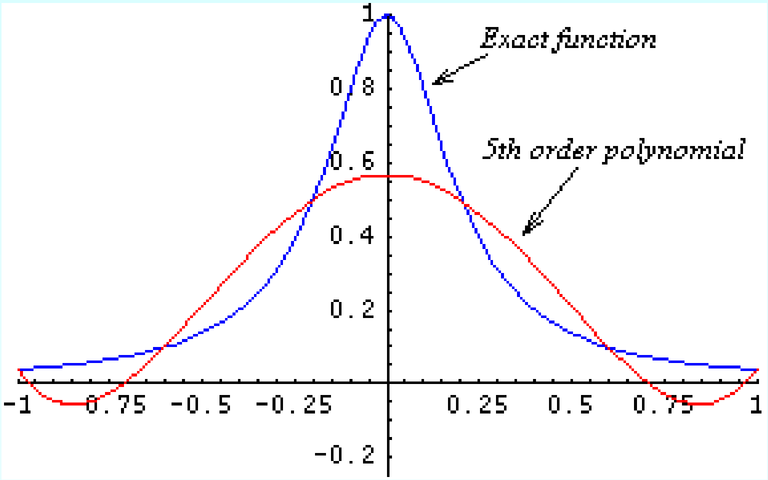



**Figure : 5th order polynomial vs. exact function**

# Why Splines ?

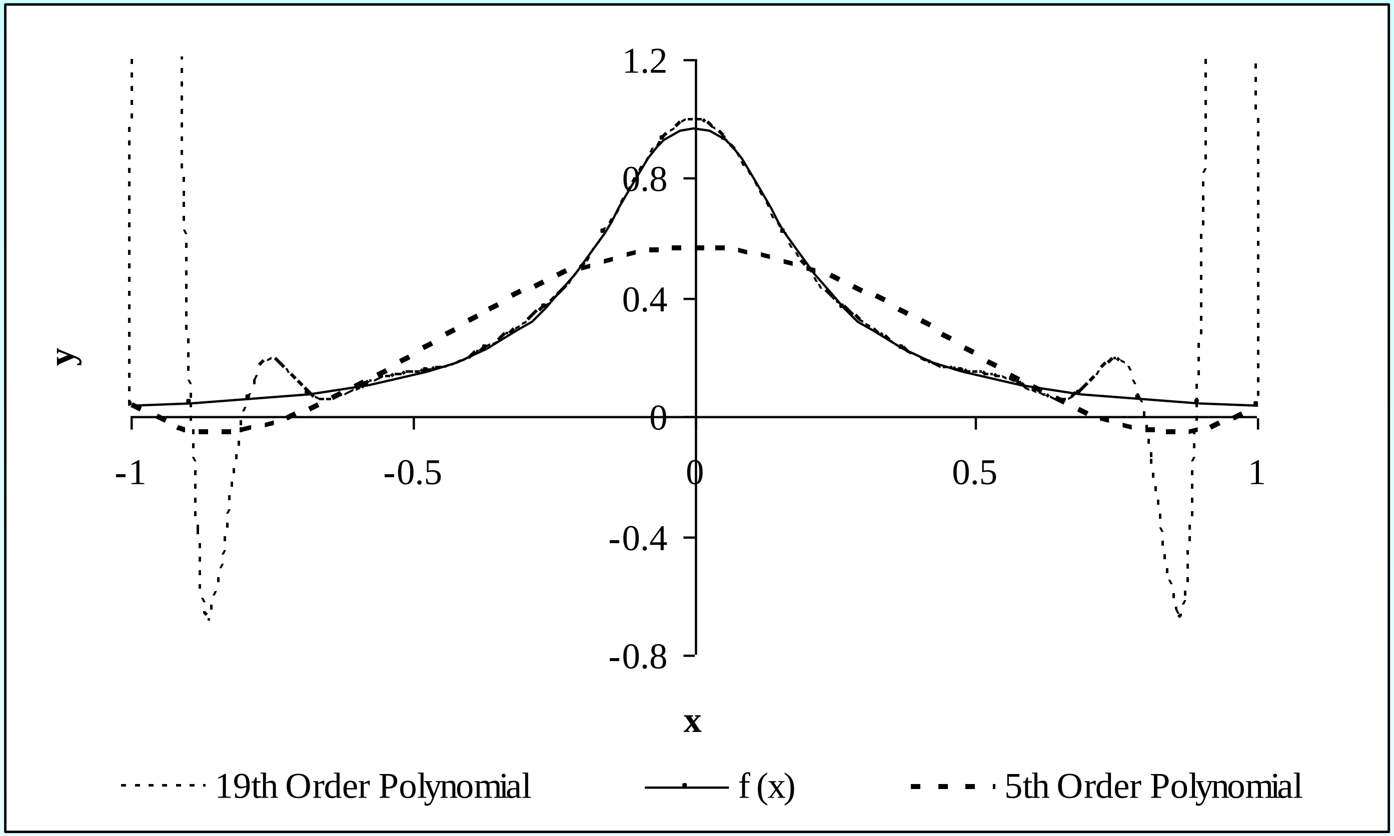



**Figure : Higher order polynomial interpolation is a bad idea**

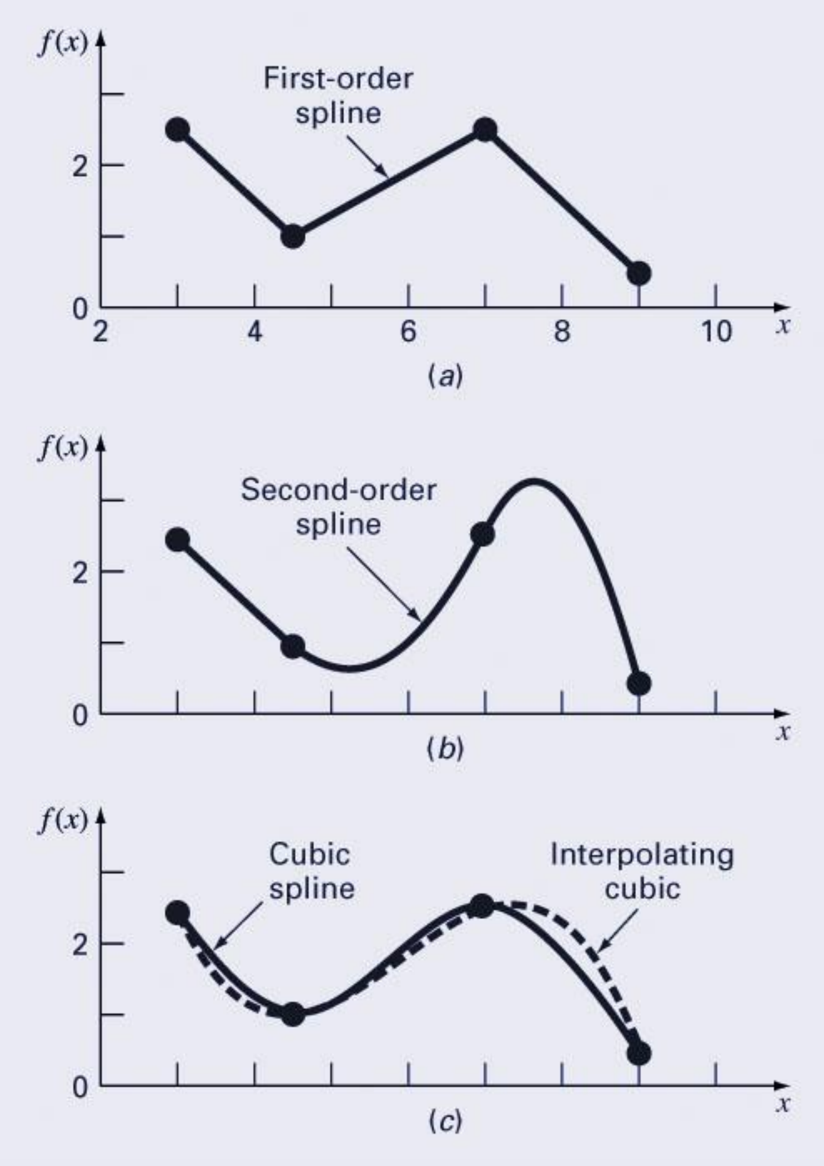
# Spline Development

a) First-order splines find straight-line equations between each pair of points that
   - Go through the points

b) Second-order splines find quadratic equations between each pair of points that
   - Go through the points
   - Match first derivatives at the interior points

c) Third-order splines find cubic equations between each pair of points that
   - Go through the points
   - Match first and second derivatives at the interior points

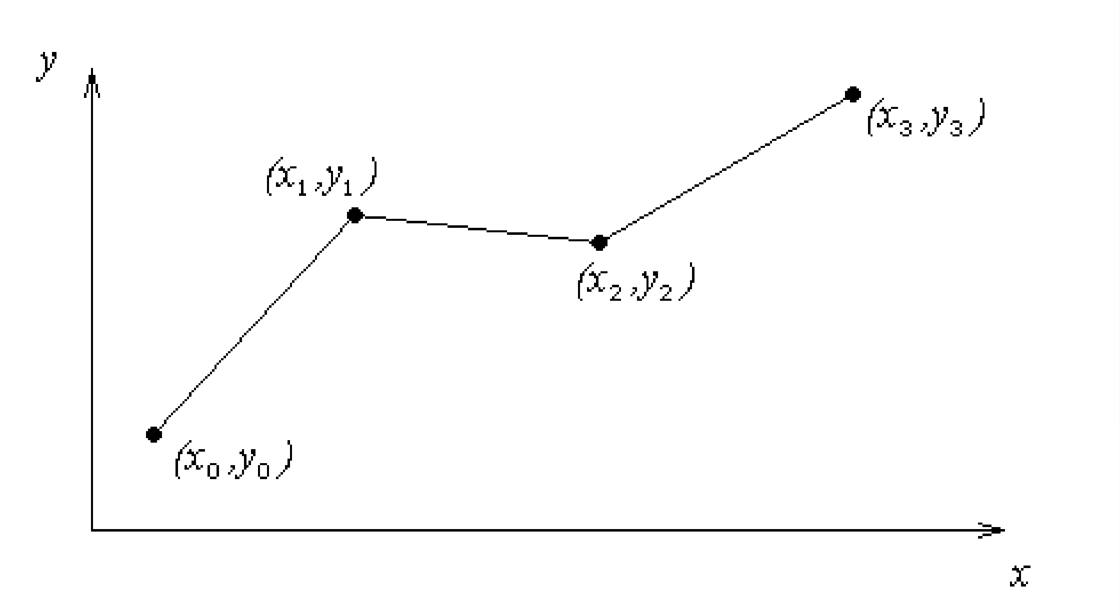
Note that the results of cubic spline interpolation are different from the results of an interpolating cubic.



# Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $\left(y_i = f(x_i)\right)$

**Figure : Linear splines**



# Linear Interpolation (contd)

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1 \\
&= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n
\end{aligned}
$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between $x_{i-1}$ and $x_i$.

# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

| $t$ (s) | $v(t)$ (m/s) |
|---|---|
| 0 | 0 |
| 10 | 227.04 |
| 15 | 362.78 |
| 20 | 517.35 |
| 22.5 | 602.97 |
| 30 | 901.67 |

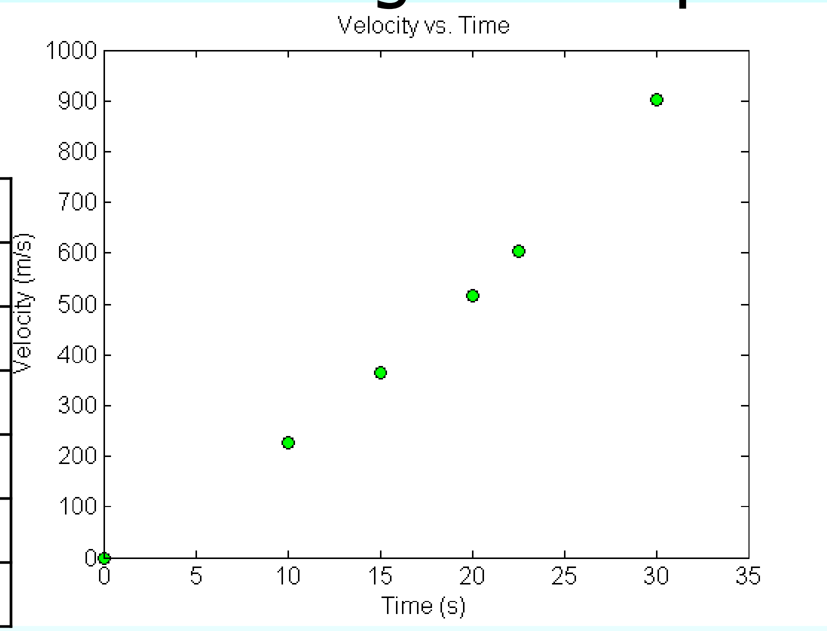



Figure. Velocity vs. time data for the rocket example

# Linear Interpolation

$$t_0 = 15, \qquad v(t_0) = 362.78$$

$$t_1 = 20, \qquad v(t_1) = 517.35$$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

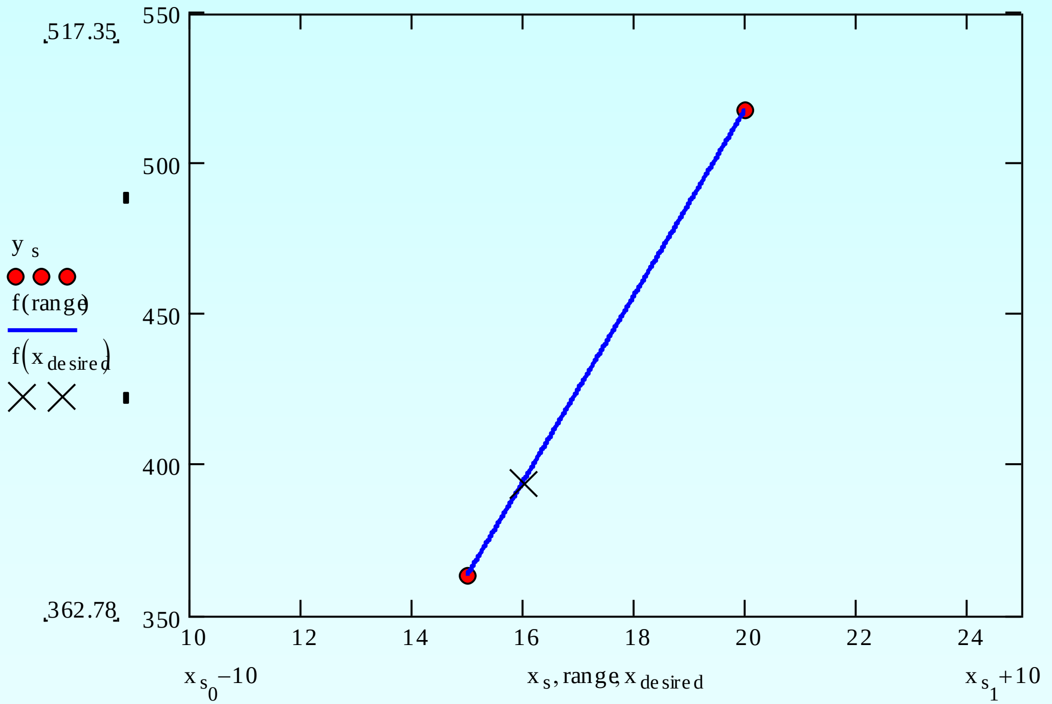
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

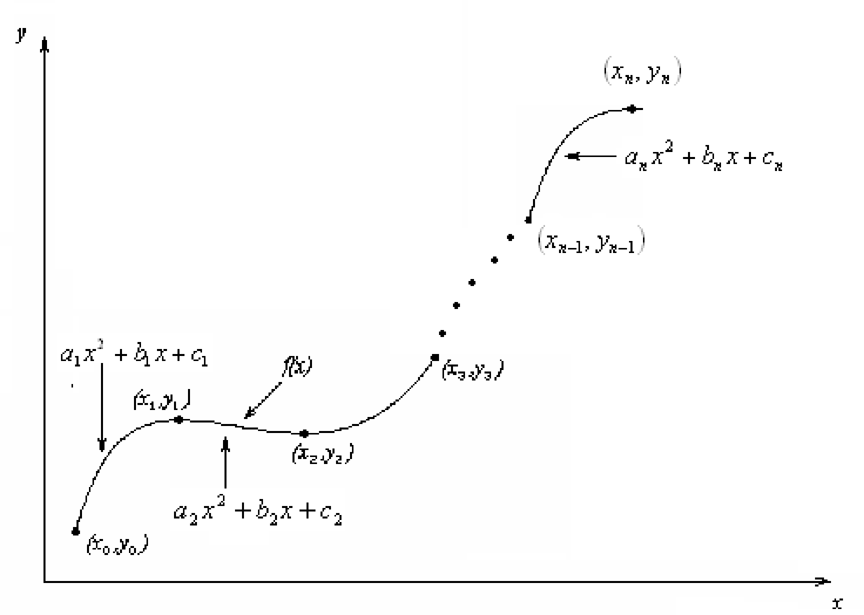
$$= 393.7 \text{ m/s}$$



# Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$




Find $a_i$, $b_i$, $c_i$, $i = 1, 2, \ldots, n$

# Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

.

.

.

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

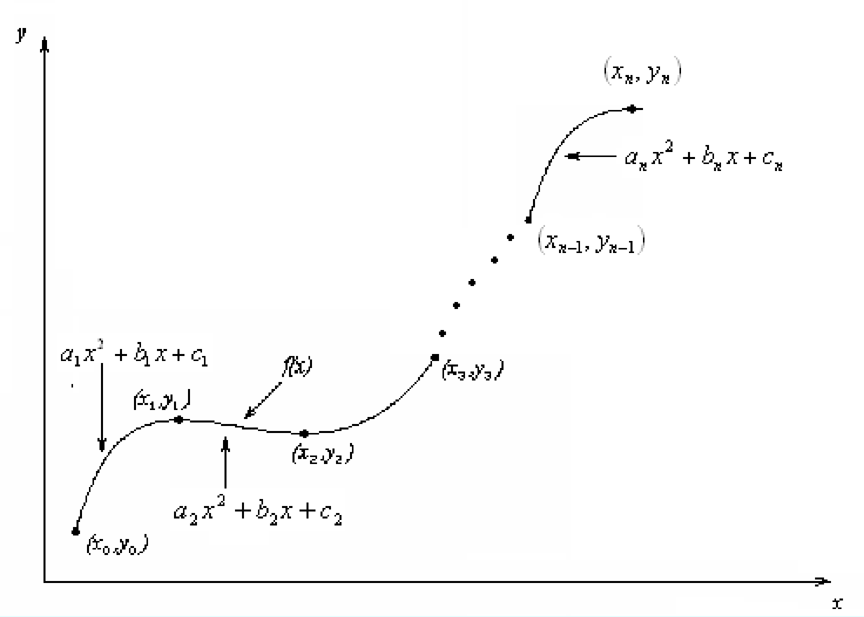
.

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$




This condition gives 2n equations

# Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

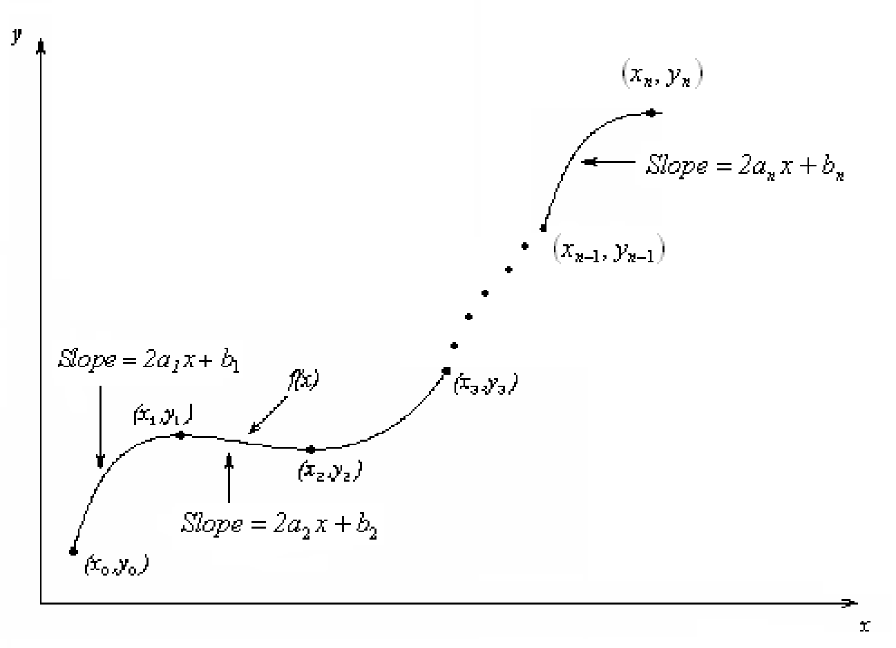
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



# Quadratic Splines (contd)

Similarly at the other interior points,

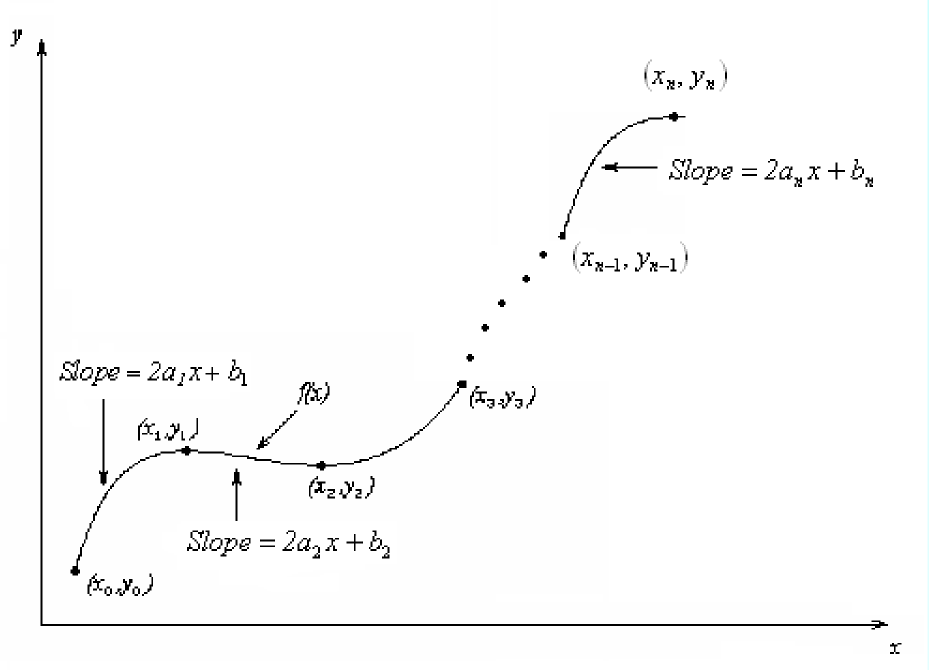
$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



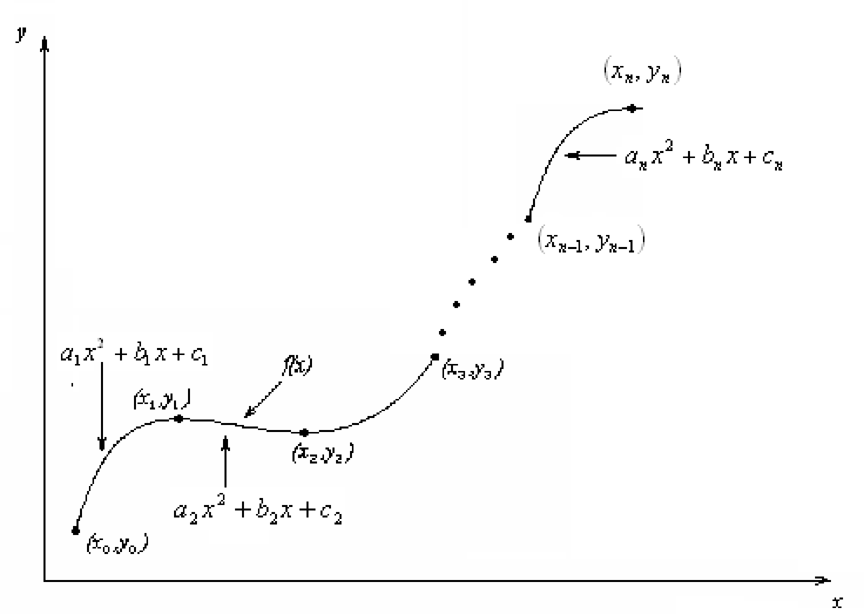

We have (n-1) such equations. The total number of equations is $(2n)+(n-1)=(3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

# Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$
\begin{aligned}
f(x) &= a_1x^2 + b_1x + c_1, && x_0 \le x \le x_1 \\
&= a_2x^2 + b_2x + c_2, && x_1 \le x \le x_2 \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&= a_nx^2 + b_nx + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



# Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table Velocity as a function of time

| $t$ (s) | $v(t)$ (m/s) |
|---|---|
| 0 | 0 |
| 10 | 227.04 |
| 15 | 362.78 |
| 20 | 517.35 |
| 22.5 | 602.97 |
| 30 | 901.67 |

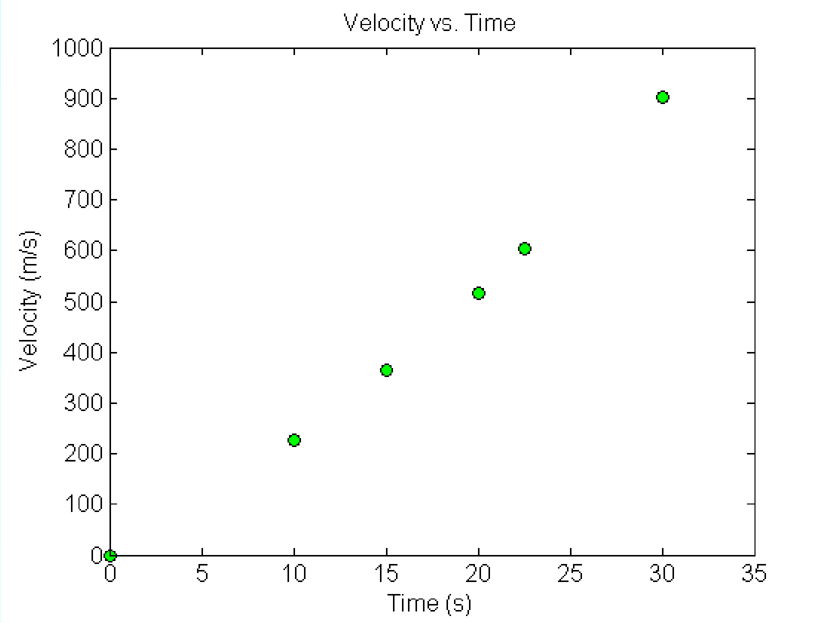



Figure. Velocity vs. time data for the rocket example

# Solution

$$
\begin{aligned}
v(t) &= a_1t^2 + b_1t + c_1, & 0 \le t \le 10 \\
&= a_2t^2 + b_2t + c_2, & 10 \le t \le 15 \\
&= a_3t^2 + b_3t + c_3, & 15 \le t \le 20 \\
&= a_4t^2 + b_4t + c_4, & 20 \le t \le 22.5 \\
&= a_5t^2 + b_5t + c_5, & 22.5 \le t \le 30
\end{aligned}
$$

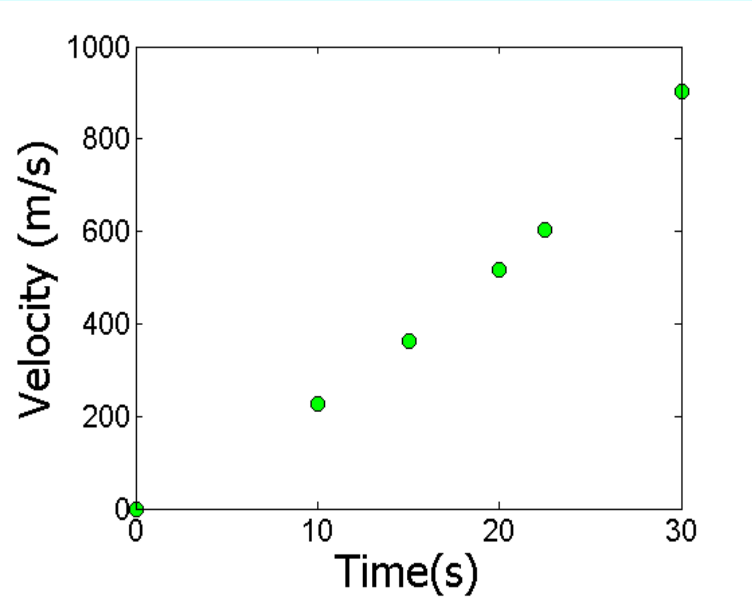
Let us set up the equations

# Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \; 0 \le t \le 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



# Each Spline Goes Through Two Consecutive Data Points

| t | v(t) |
|---|---|
| **s** | **m/s** |
| 0 | 0 |
| 10 | 227.04 |
| 15 | 362.78 |
| 20 | 517.35 |
| 22.5 | 602.97 |
| 30 | 901.67 |

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

# Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1,\ 0 \le t \le 10$$

$$= a_2 t^2 + b_2 t + c_2, 10 \le t \le 15$$

$$\left.\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\right|_{t=10} = \left.\frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\right|_{t=10}$$

$$\left.\left(2a_1 t + b_1\right)\right|_{t=10} = \left.\left(2a_2 t + b_2\right)\right|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

# Derivatives are continuous at Interior Data Points

At t=10

$$2a_1(10)+b_1-2a_2(10)-b_2=0$$

At t=15

$$2a_2(15)+b_2-2a_3(15)-b_3=0$$

At t=20

$$2a_3(20)+b_3-2a_4(20)-b_4=0$$

At t=22.5

$$2a_4(22.5)+b_4-2a_5(22.5)-b_5=0$$

# Last Equation

$$a_1 = 0$$

# Final Set of Equations

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

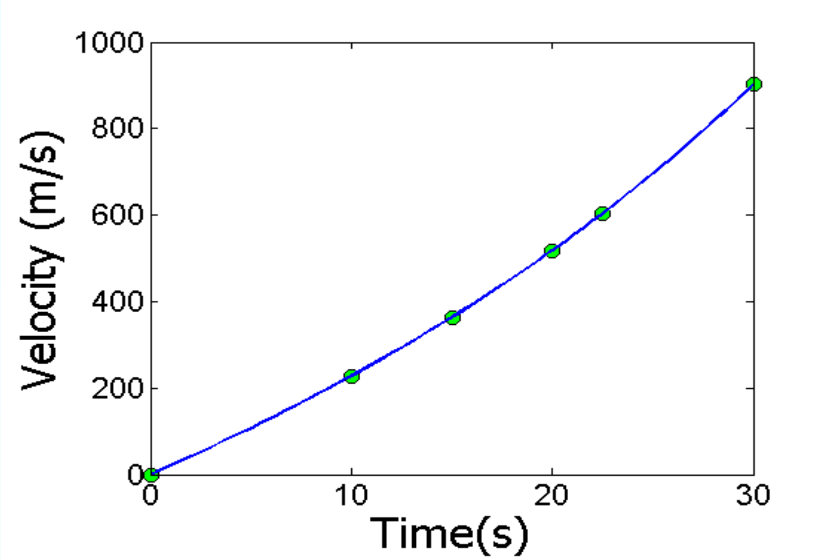
# Coefficients of Spline

| $i$ | $a_i$ | $b_i$ | $c_i$ |
|---|---|---|---|
| 1 | 0 | 22.704 | 0 |
| 2 | 0.8888 | 4.928 | 88.88 |
| 3 | −0.1356 | 35.66 | −141.61 |
| 4 | 1.6048 | −33.956 | 554.55 |
| 5 | 0.20889 | 28.86 | −152.13 |

# Quadratic Spline Interpolation
# Part 2 of 2

# Final Solution

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$



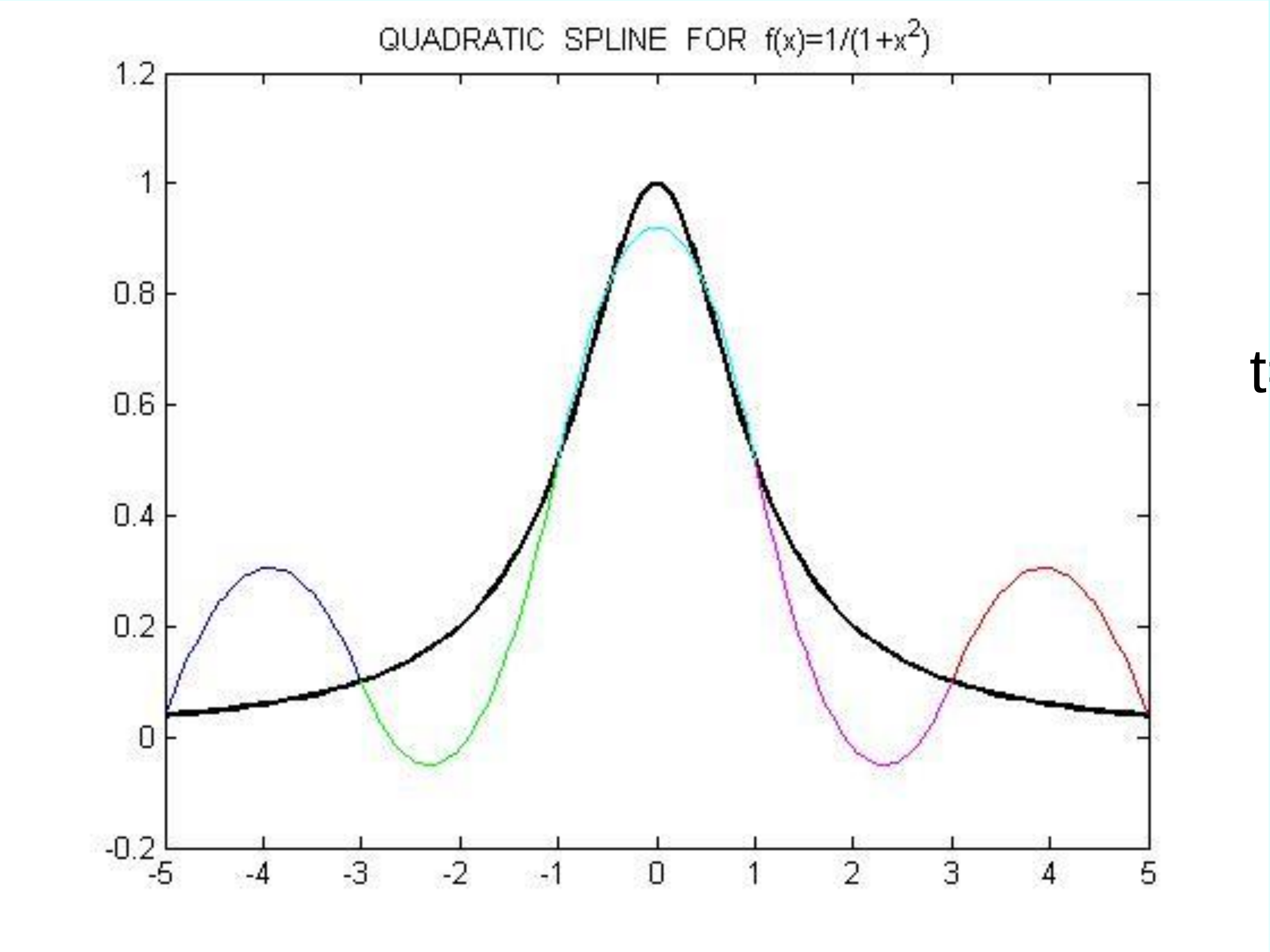
# Velocity at a Particular Point

a) Velocity at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

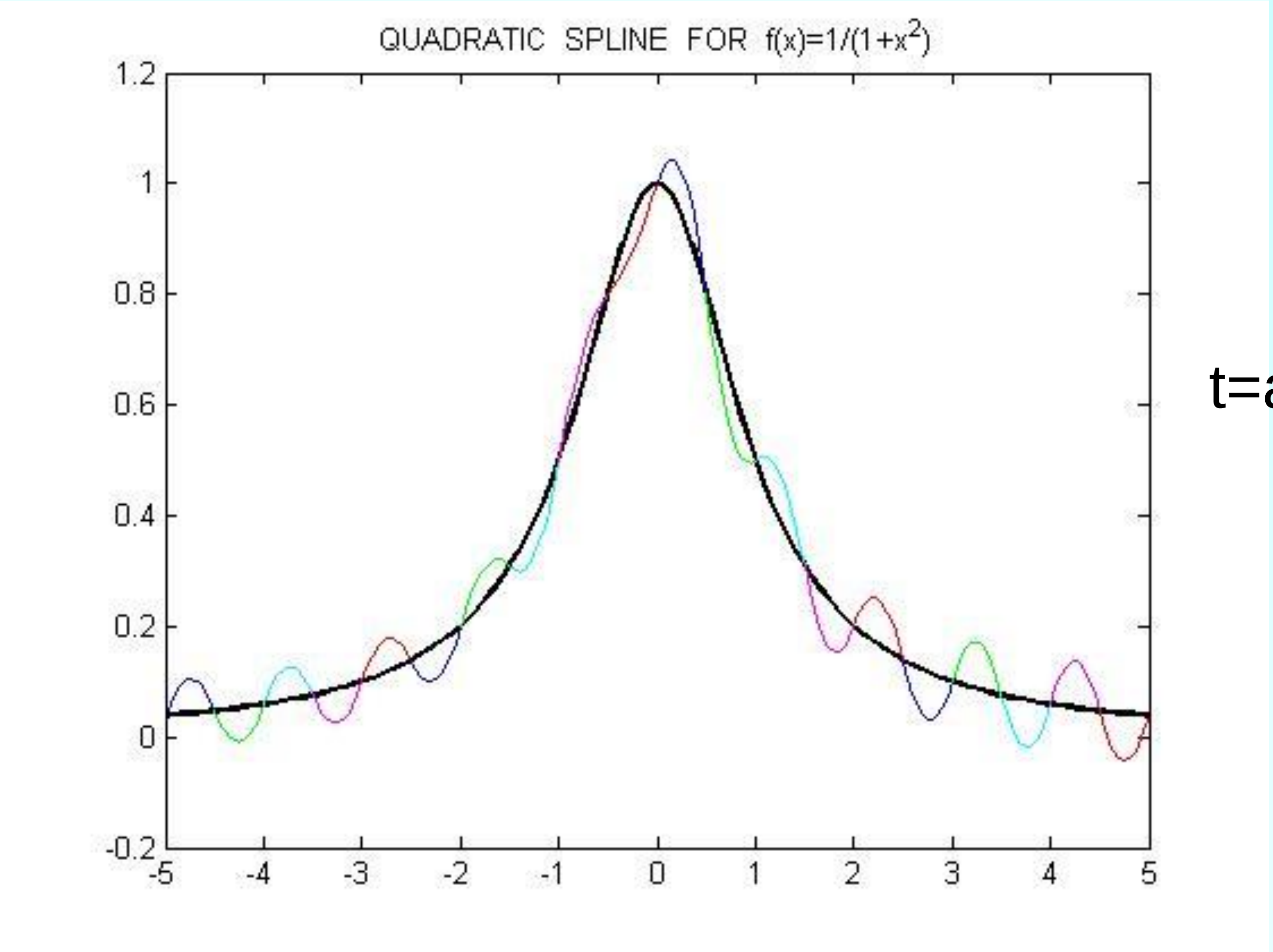
$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

# Quadratic Spline Graph




t=a:2:b;

# Quadratic Spline Graph




t=a:0.5:b;

# Natural Cubic Spline Interpolation

**SPLINE OF DEGREE k = 3**

- The domain of S is an interval [a,b].
- S, S′, S″ are all continuous functions on [a,b].
- There are points $t_i$ (the knots of S) such that a = $t_0 < t_1 < .. t_n = b$ and such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$.

| | | | | |
|---|---|---|---|---|
| x | $t_0$ | $t_1$ | … | $t_n$ |
| y | $y_0$ | $y_1$ | … | $y_n$ |

*$t_i$ are knots*

# Natural Cubic Spline Interpolation

$$S(x) = \begin{cases} S_0(x), \ x \in [x_0, x_1] \\ S_1(x), \ x \in [x_1, x_2] \\ \cdots \\ S_{n-1}(x), \ x \in [x_{n-1}, x_n] \end{cases}$$

$S_i(x)$ *is a cubic polynomial that will be used on the subinterval [* $\boldsymbol{x_i, x_{i+1}}$ *].*

# Natural Cubic Spline Interpolation

- $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$
  - 4 Coefficients with n subintervals = 4n equations
  - There are $4_{n-2}$ conditions
    - Interpolation conditions
    - Continuity conditions

    - Natural Conditions
  - $S''(x_0) = 0$
  - $S''(x_n) = 0$