Numerical Analysis with Computer Applications

BE(Chem. Engg.)
15 batch

Topic: Interpolation

Finite Difference Operators

Newton's Forward Difference Interpolation Formula

Newton's Backward Difference Interpolation Formula

Lagrange's Interpolation Formula

Divided Differences

Interpolation in Two Dimensions

Cubic Spline Interpolation

Introduction

Finite differences play an important role in numerical techniques, where tabulated values of the functions are available.
For instance, consider a function

$$y = f(x).$$

As *x* takes values

$$x_0, x_1, x_2, \ldots, x_n,$$

let the corresponding values of *y* be

$$y_0, y_1, y_2, \ldots, y_n.$$

That is, for given a table of values,

$$(x_k, y_k), k = 0, 1, 2, \ldots, n;$$

the process of estimating the value of *y*, for any intermediate value of *x*, is called interpolation.

- **The method of computing the value of *y*, for a given value of *x*, lying outside the table of values of *x* is known as extrapolation.**

- **If the function $f(x)$ is known, the value of y corresponding to any x can be readily computed to the desired accuracy.**

- **For interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.**

Finite Difference Operators

Finite Difference Operators

Forward Differences

Backward Differences

Central Difference

Forward Differences

For a given table of values

$$(x_k, y_k), k = 0, 1, 2, \ldots, n$$

with equally spaced abscissas of a function $y = f(x)$,

we define the forward difference operator Δ as follows

$$\Delta y_i = y_{i+1} - y_i, \qquad i = 0, 1, \ldots, (n-1)$$

To be explicit, we write

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
&\vdots \qquad \vdots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

These differences are called ***first differences of the function y*** and are denoted by the symbol Δy_i

Here, Δ is called the first difference operator

- **Similarly, the differences of the first differences are called second differences, defined by**

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \qquad \Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

Thus, in general

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

Here Δ^2 is called the ***second difference*** operator. Thus, continuing, we can define, ***r-th*** difference of ***y***, as

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$

- By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values

$$(x_k, y_k), k = 0, 1, \ldots, 6$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0	Δy_0					
x_1	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$	$\Delta^5 y_0$	
x_3	y_3	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_2$	$\Delta^4 y_1$	$\Delta^5 y_1$	$\Delta^6 y_0$
x_4	y_4	Δy_4	$\Delta^2 y_3$	$\Delta^3 y_3$			
x_5	y_5	Δy_5	$\Delta^2 y_4$				
x_6	y_6						

- **This difference table is called *forward difference* table or *diagonal difference* table. Here, each difference is located in its appropriate column, midway between the elements of the previous column.**

- **Please note that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the leading term, while the differences**

$$\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$$

- **are called leading differences**

Example

Construct a forward difference table for the following values of *x* and *y*:

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.37	0.518	0.697

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
0.3	0.067	0.064	0.017			
0.5	0.148	0.081	0.019	0.002		
0.7	0.248	0.1	0.022	0.003	0.001	
0.9	0.37	0.122	0.026	0.004	0.001	0
1.1	0.518	0.148	0.031	0.005	0.001	0
1.3	0.697	0.179				

Example

Express $\Delta^2 y_0$ and $\Delta^3 y_0$ in terms of the values of the function *y*.

Solution:

Noting that each higher order difference is defined in terms of the lower order difference, we have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

and

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta_{y1} - \Delta y_0)$$
$$= (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$

Hence, we observe that the coefficients of the values of *y*, in the expansion of $\Delta^2 y_0, \Delta^3 y_0$, are binomial coefficients.

Thus, in general, we arrive at the following result: -

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - {}^nC_3 y_{n-3} + \cdots + (-1)^n y_0$$

Example

Show that the value of y_n can be expressed in terms of the leading value y_0 and the leading differences

$$\Delta y_0, \Delta^2 y_0, \ldots, \Delta^n y_0.$$

Solution

The forward difference table will be

$$\left.\begin{array}{lll} y_1 - y_0 = \Delta y_0 & \text{or} & y_1 = y_0 + \Delta y_0 \\ y_2 - y_1 = \Delta y_1 & \text{or} & y_2 = y_1 + \Delta y_1 \\ y_3 - y_2 = \Delta y_2 & \text{or} & y_3 = y_2 + \Delta y_2 \end{array}\right\}$$

Similarly,

$$\left.\begin{array}{l}\Delta y_1 - \Delta y_0 = \Delta^2 y_0 \quad \text{or} \quad \Delta y_1 = \Delta y_0 + \Delta^2 y_0 \\ \Delta y_2 - \Delta y_1 = \Delta^2 y_1 \quad \text{or} \quad \Delta y_2 = \Delta y_1 + \Delta^2 y_1\end{array}\right\}$$

Similarly, we can also write

$$\left.\begin{array}{l}\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0 \quad \text{or} \quad \Delta^2 y_1 = \Delta^2 y_0 + \Delta^3 y_0 \\ \Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1 \quad \text{or} \quad \Delta^2 y_2 = \Delta^2 y_1 + \Delta^3 y_1\end{array}\right\}$$

$$\Delta y_2 = (\Delta y_0 + \Delta^2 y_0) + (\Delta^2 y_0 + \Delta^3 y_0)$$

$$= \Delta y_0 + 2\Delta^2 y_0 + \Delta^3 y_0$$

$$y_3 = y_2 + \Delta y_2$$

$$= (y_1 + \Delta y_1) + (\Delta y_1 + \Delta^2 y_1)$$

$$= y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0$$

$$= (1 + \Delta)^3 y_0$$

Similarly, we can symbolically write

$$y_1 = (1+\Delta) y_0,$$

$$y_2 = (1+\Delta)^2 y_0,$$

$$y_3 = (1+\Delta)^3 y_0$$

$$\cdots\cdots\cdots$$

$$y_n = (1+\Delta)^n y_0$$

Hence, we obtain

$$y_n = y_0 + {}^nC_1 \Delta y_0 + {}^nC_2 \Delta^2 y_0 + {}^nC_3 \Delta^3 y_0 + \cdots + \Delta^n y_0$$

$$y_n = \sum_{i=0}^{n} {}^nC_i \Delta^i y_0$$

Backward Differences

For a given table of values

$$(x_k, y_k), k = 0, 1, 2, \ldots, n$$

of a function $y = f(x)$ with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1} i = n, \quad (n-1), \ldots, 1$$

OR

$$\begin{aligned}
\nabla y_1 &= y_1 - y_0 \\
\nabla y_2 &= y_2 - y_1 \\
\vdots \quad & \quad \vdots \quad \vdots \\
\nabla y_n &= y_n - y_{n-1}
\end{aligned}$$

The differences of these differences are called *second differences* and they are denoted by

$$\nabla^2 y_2, \nabla^2 y_3, \ldots, \nabla^2 y_n.$$

That is

$$\begin{aligned}
\nabla^2 y_1 &= \nabla y_2 - \nabla y_1 \\
\nabla^2 y_2 &= \nabla y_3 - \nabla y_2 \\
\vdots \quad & \quad \vdots \qquad \quad \vdots \\
\nabla^2 y_n &= \nabla y_n - \nabla y_{n-1}
\end{aligned}$$

Thus, in general, the second backward differences are

$$\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \quad i = n, (n-1), \ldots, 2$$

while the *k-th* backward differences are given as

$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}, \quad i = n, (n-1), \ldots, k$$

These backward differences can be systematically arranged for a table of values

$$(x_k, y_k), k = 0, 1, \ldots, 6$$

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
x_1	y_1	∇y_1	$\nabla^2 y_2$	$\nabla^3 y_3$	$\nabla^4 y_4$		
x_2	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_4$	$\nabla^4 y_5$	$\nabla^5 y_5$	
x_3	y_3	∇y_3	$\nabla^2 y_4$	$\nabla^3 y_5$	$\nabla^4 y_6$	$\nabla^5 y_6$	$\nabla^6 y_6$
x_4	y_4	∇y_4	$\nabla^2 y_5$	$\nabla^3 y_6$			
x_5	y_5	∇y_5	$\nabla^2 y_6$				
x_6	y_6	∇y_6					

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Example

Show that any value of y can be expressed in terms of y_n and its backward differences.

Solution

From $\nabla y_i = y_i - y_{i-1} i = n, \quad (n-1), \ldots, 1$

We get $y_{n-1} = y_n - \nabla y_n$ $\quad$ $y_{n-2} = y_{n-1} - \nabla y_{n-1}$

From $\nabla^2 y_i = \nabla y_i - \nabla y_{i-1}, \quad i = n, (n-1), \ldots, 2$

We get $\nabla y_{n-1} = \nabla y_n - \nabla^2 y_n$

From these equations, we obtain

$$y_{n-2} = y_n - 2\nabla y_n + \nabla^2 y_n$$

Similarly, we can show that

$$y_{n-3} = y_n - 3\nabla y_n + 3\nabla^2 y_n - \nabla^3 y_n$$

Symbolically, these results can be rewritten as follows:

$$y_{n-1} = (1-\nabla)y_n$$

$$y_{n-2} = (1-\nabla)^2 y_n$$

$$y_{n-3} = (1-\nabla)^3 y_n$$

$$\cdots\cdots\cdots$$

$$y_{n-r} = (1-\nabla)^r y_n$$

$$y_{n-r} = y_n - {}^nC_1 \nabla y_n + {}^nC_2 \nabla^2 y_n - \cdots + (-1)^r \nabla^r y_n$$

Central Differences

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts

$$\delta y_{1/2} = y_1 - y_0, \qquad \delta y_{3/2} = y_2 - y_1,$$

In general

$$\delta y_i = y_{i+(1/2)} - y_{i-(1/2)}$$

Higher order differences are defined as follows:

$$\delta^2 y_i = \delta y_{i+(1/2)} - \delta y_{i-(1/2)}$$

$$\delta^n y_i = \delta^{n-1} y_{i+(1/2)} - \delta^{n-1} y_{i-(1/2)}$$

These central differences can be systematically arranged as indicated in the Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
x_1	y_1	$\delta y_{1/2}$	$\delta^2 y_1$	$\delta^3 y_{3/2}$	$\delta^4 y_2$		
x_2	y_2	$\delta y_{3/2}$	$\delta^2 y_2$	$\delta^3 y_{5/2}$	$\delta^4 y_3$	$\delta^5 y_{5/2}$	
x_3	y_3	$\delta y_{5/2}$	$\delta^2 y_3$	$\delta^3 y_{7/2}$	$\delta^4 y_4$	$\delta^5 y_{7/2}$	$\delta^6 y_3$
x_4	y_4	$\delta y_{7/2}$	$\delta^2 y_4$	$\delta^3 y_{9/2}$			
x_5	y_5	$\delta y_{9/2}$	$\delta^2 y_5$				
x_6	y_6	$\delta y_{11/2}$					

Thus, we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x.

Suppose, we are given consecutive values of *x* differing by *h* say x, $x + h$, $x + 2h$, $x + 3h$, etc. The corresponding values of *y* are $y_x, y_{x+h}, y_{x+2h}, y_{x+3h},$

As before, we can form the differences of these values.

Thus

$$\Delta y_x = y_{x+h} - y_x = f(x+h) - f(x)$$

$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly

$$\nabla y_x = y_x - y_{x-h} = f(x) - f(x-h)$$

$$\delta y_x = y_{x+(h/2)} - y_{x-(h/2)} = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$