Coarse-grid-correction preconditioner for the Helmholtz Equation

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Many wave phenomena are well described by the wave equation. When the considered wave has a fixed frequency the wave equation is mostly re-written in the frequency domain which results in the Helmholtz equation

$$-\Delta u(\mathbf{x}) - k^2(\mathbf{x}) = g(\mathbf{x}). \tag{1}$$

It is also possible to approximate the time domain solution with a summation of solutions for several frequencies. Applications consist of the propagation of sound, sonar, seismic, and many more. We emphasize on the seismic imaging used for searching oil and gas in the subsoil. In order to have a good image of the under ground, often high frequencies are chosen for high resolution. The discrete analogue of the Helmholtz Equation (1) is a combination of a symmetric positive definite matrix (Poisson) and the mass matrix i.e.

$$L + iC - M = g \tag{2}$$

where C represents the boundary conditions.
The discretized linear System (2) has two characteristic properties :

- the product of the wave number and the step size should be smaller than a given constant,
- if the wavenumber increases the operator has more and more negative eigenvalues.

Solving the discretized Helmholtz equation have been a challenging problem. Krylov methods with classical preconditioners and Multigrid methods tend to break down due to high indefiniteness for high wavenumber problems. During the year 2005, the idea of using complex shifted Laplacian as preconditioner (CSLP) [1] gave rise to fast and robust Krylov solvers for Helmholtz. It appears that the amount of work increases linearly with the wavenumber. This happens as the near kernel components tend to appear more frequently as the wavenumber increases.

The combination of the complex shifted Laplacian with a multigrid deflation technique was first proposed in [2] and later analyzed in [3]. In these works the shifted Laplacian is attributed the role of a multigrid smoother. Where as the coarse grid correction (CGC) is performed as Preconditioner to the outer Krylov iterations. We investigate several CGC techniques that differ in the choice of the coarse grid operator. A rigorous Fourier mode analysis for the one-dimensional problem with Dirichlet boundary conditions is performed to

Frequency	Solve Time	Solve Time	Iterations	Iterations
	SLP-F	**ADEF1-F**	**SLP-F**	**ADEF1-F**
$f = 1$	1.23	5.08	13	7
$f = 10$	40.01	21.83	106	8
$f = 20$	280.08	131.30	177	12
$f = 40$	20232.6	3997.7	340	21

Table 1: SLP and ADEF1 performance comparison for Marmousi problem.

distinguish these different techniques based on different coarse grid operator. This creates opportunity to optimize the coarse grid correction preconditioner. The CGC technique combined with CSLP has been implemented in multilevel fashion, similar to that of multigrid in Petsc. We refer this combination of CSLP and CGC techniques as ADEF1 preconditioner. Numerical results for two-dimensional and three-dimensional problems show significant speed up in comparison with CSLP and other preconditioners. The iteration count remains constant for medium wavenumbers and increases mildly for high wavenumber at the application cost of CGC. However one can notice that the proposed deflation preconditioner pays off and which is illustrated by a gain in solve time for industrial problems. Such evidence is presented in Table (1), where a brief comparison of iteration and solve time is presented in Table for CSLP and ADEF1 preconditioners.

References:

[1] Y.A. Erlangga and C.W. Oosterlee and C. Vuik A Novel Multigrid Based Preconditioner For Heterogeneous Helmholtz Problems SIAM J. Sci. Comput.,27, pp. 1471-1492, 2006

[2] Y. A. Erlangga and R. Nabben, On a Multilevel Krylov Method for the Helmholtz Equation Preconditioned by Shifted Laplacian, ETNA, pp. 403-424, 31, 2008.

[3] A.H. Sheikh and D. Lahaye and C. Vuik On the convergence of shifted Laplace preconditioner combined with multilevel deflation Numerical Linear Algebra with Applications, 20, pp. 645-662, 2013 http://ta.twi.tudelft.nl/nw/users/vuik/papers/She13LV.pdf