Fast iterative schemes for numerical solution of Helmholtz equation

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Helmholtz Problem

Abstract: We consider iterative schemes for solution of Helmholtz equation. To solve complex and nonsymmetric linear system of Helmholtz equation, GMRES is preconditioned with complex shifted Laplace preconditioner. For larger wavenumber, the system have some eigenvalues near zero, which cause slow convergence. Deflation technique is used to fix this.

Model Problem: The Helmholtz equation, which is used to model waveproblems in various areas, is given as

$$-\Delta \mathbf{u}(x,y) - k^2 \mathbf{u}(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \tag{1}$$

with Dirichlet

$$u(x,y) = f \tag{2}$$

and Sommerfeld boundary conditions

$$(\frac{\delta u}{\delta n} - \imath k u) = 0 \tag{3}$$

where u is physical variable, $k = \frac{2\pi}{\lambda} = \frac{\omega}{c(x)}$ is wavenumber and g is source function.

Discretization and Linear System: Finite difference method is used to obtained discrete analogue of system. Two different strategies are required to discretize Dirichlet and Sommerfeld (Neumann) conditions. Linear system obtained from discretization with Sommerfeld conditions is complex valued and nonsymmetric, restricting the selection of iterative methods. For larger wavenumber in problem, we need to use high resolution of mesh. The system has negative real part of eigenvalues for sufficiently large wavenumber k.

Iterative Schemes

The Linear system $Au = g$ is nonsymmetric and indefinite (negative real part of eigenvalues), restricting the selection of iterative methods. System is solved with GMRES preconditioned with ILU(0) and ILU(*tol*), which is costly in terms of storeage, and shifted Laplace preconditioner $M(\beta_1 = 0, \beta_2 = 1)$.

Shifted Laplace Preconditioner: The shifted Laplace preconditioner is obtained by discretization of

$$-\Delta \mathbf{u} - (\beta_1 + \imath\beta_2)k^2 \mathbf{u}(x,y)$$

on same boundaries of problem, where β_1 and β_2 are real and imaginary shifts respectively. Preconditioning matrix is SPD and more diagonally dominant then of the coefficient matrix A. Amongst all these, shifted Laplace is chosen.

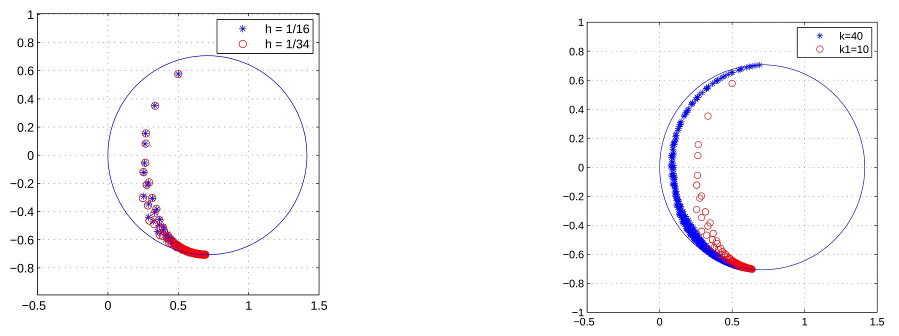

Figure 1: (a) Spectrum for Helmholtz Operator preconditioned with shifted Laplace preconditioner $M(0,1)$ different mesh sizes $k = 10$ (b)Spectrum for different wave numbers $k = 10$ and $k = 40$

Deflation

For large wavenumber k, some eigenvalues are near to zero, which cause slow convergence.To fix this problem *Deflation* is used. Deflation preconditioner P is

$$P = I - AQ$$

$$\text{with} \quad Q = ZE^{-1}Z^T \quad E = Z^T AZ$$

Z is matrix of deflating vectors, chosen such that E is invertible. In deflation, eigenvectors corresponding to unwanted eigenvalues are projected to minimum eigenvalue, making their contribution disappear in Krylov basis.

Numerical Experiments

Experiments with deflation are done with Helmholtz equation with Dirichlet conditions, restricting to SPD and residuals are used as deflation vectors. Two graphs distinguish working of deflation used in without preconditioner and with preconditioner schemes.

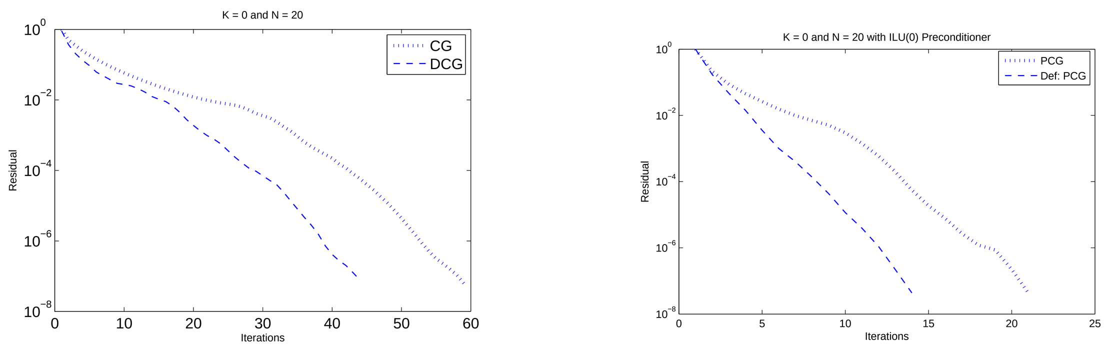

Figure 2: (a) Convergence History for PCG and Deflated PCG for (b) Convergence History of PCG and Deflated PCG for Spectrum for $N = 20$ and $k = 10$. ILU is used as preconditioner

Conclusive remarks and further work

For the linear system obtained from discretization of Helmholtz equation by the FDM, GMRES preconditioned with shifted Laplace preconditioner is best choice. For small wavenumber k, ILU preconditioners works, but they are no more of use for larger k, we see storage problem. With shifted Laplace preconditioner, system have some small eigen near zero 0, causing GMRES convergence slow. This problem is going to be serious even for increasing k, but can be fixed by deflation, by taking approximation to eigenvectors as deflation vectors. Approximated eigenvectors as deflation vectors for equation (1) alongwith some iterative solver for Preconditioner will rise to an efficient iterative scheme for equation (1).

References

- ERLANGGA. *A robust and efficient iterative method for numerical solution of Helmholtz equation.*, PhD Thesis at the Delft University of Technology, 2005
- TANG. *Two level Preconditioned Conjugate Gradient Methods with Application to Bubbly flow problems.* PhD Thesis at Delft University of Technology, 2008.
- ERLANGGA. *On a multilevel Krylov Method for the Helmholtz equation preconditioned by shifted Laplacian.* Unpublished, 2008.

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