On iterative solutions of Helmholtz equation

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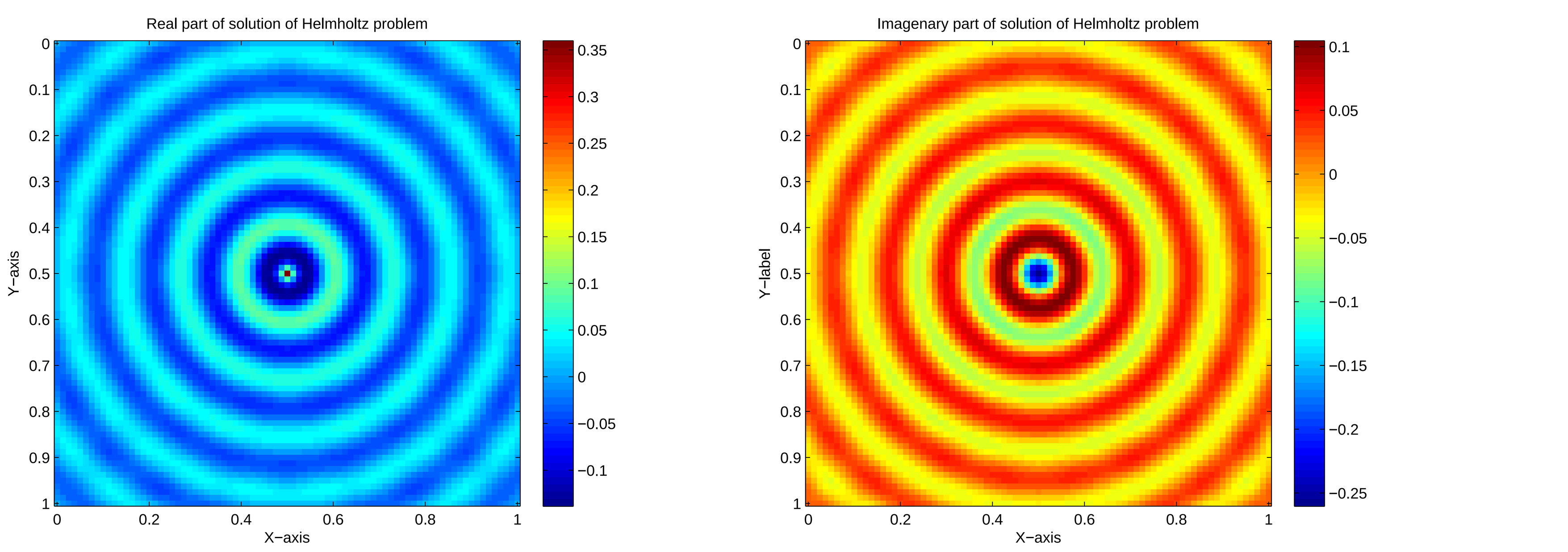

Figure 1: Real *(left)* part and imagenary *(right)* part of solution of the Helmholtz equation solved by GMRES preconditioned with shifted Laplace preconditioner $M(1, 0.1)$

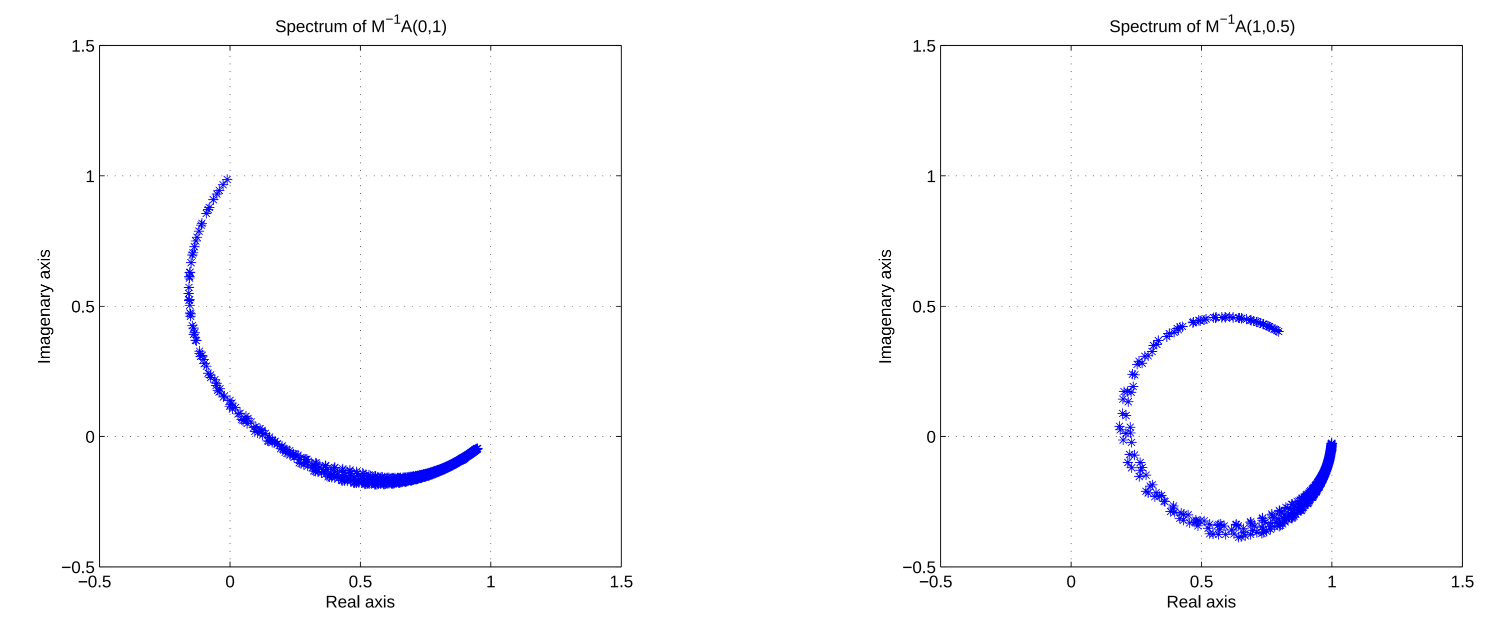

Figure 2: Spectrum for preconditioned Helmholtz Operator preconditioned with shifted Laplace preconditionerfig (a) $M(0, 1)$ (b) $M(1, 0.5)$

Helmholtz Model Problem

The Helmholtz equation is

$$-\Delta \mathbf{u}(x, y) - k^2 \mathbf{u}(x, y)\mathbf{u}(x, y) = \mathbf{g}(x, y) \tag{1}$$

with Sommerfeld boundary conditions

$$(\frac{\delta u}{\delta n} - \imath k u) = 0 \tag{2}$$

where

- $\frac{\delta u}{\delta n}$ is normal derivative of u
- u is physical variable,
- $k = \frac{2\pi}{\lambda} = \frac{\omega}{c(x)}$ is wavenumber and
- g is source function.

Discretization Finite difference method :

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & k^2 + 5h^2 & -1 \\ & -1 & \end{bmatrix} u_{i,j} = g_{i,j}$$

Linear system $Au = g$

⋆ Symmetric

⋆ Complex valued

⋆ Indefiniteness for sufficient large k

Iterative Schemes

Indefiniteness of the linear system $Au = g$ leaves less choice, still best are **Krylov subspace methods**, and **GMRES**, in particular.

Shifted Laplace Preconditioner:

$$-\Delta \mathbf{u} - (\beta_1 + \imath\beta_2)k^2 \mathbf{u}(x, y) \tag{3}$$

- with same boundary conditions
- spectrum of preconditioned system is bounded within unit circle
- outlayers of spectrum rushes to zero as k increases

• *Multigrid works well for SL preconditioner than Helmholtz , because of inclusion of imagenary shift*

Deflation

$$P = I - AQ \tag{4}$$

$$\text{with} \quad Q = ZE^{-1}Z^T \quad E = Z^T AZ \tag{5}$$

Z is deflation matrix.
In Multigrid Matrix, $Z = R$, *Restriction* operator .

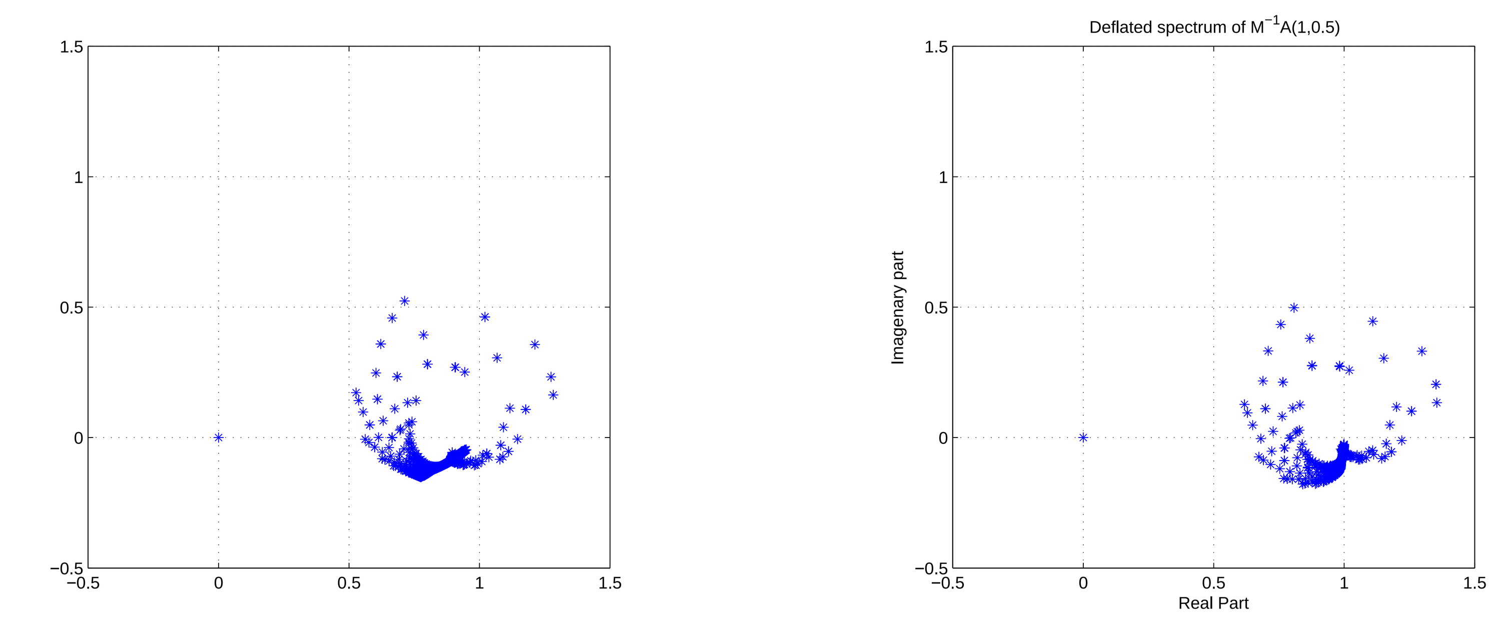

Figure 3: Deflated spectrum of $M^{-1}A$ with left: $M = M(0, 1)$ and right: $M = M(1, 0.5)$

Numerical Experiments

k	SLP	SLP with deflation
10	10	7
20	19	9
30	30	11
40	40	13
50	51	15

⋆ **Also h-independent scheme.**

Conclusive remarks and further work

- sparse deflation (Multigrid deflation)
- h-independent solution

Further:

- Analysis of scheme *(Local Fourier Analysis, in process)*
- Proceeding to Wedge problem.

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