On iterative solutions of Helmholtz equation

A. H. Sheikh, Kees Vuik and Domenico Lahaye

Helmholtz Model Problem

The Helmholtz equation is

\[-\Delta \mathbf{u}(x, y) - k^2 \mathbf{u}(x, y) = \mathbf{g}(x, y)\]  

with Sommerfeld boundary conditions

\[\frac{\partial \mathbf{u}}{\partial n} - iku = 0\]

where

- \(\frac{\partial \mathbf{u}}{\partial n}\) is normal derivative of \(\mathbf{u}\)
- \(\mathbf{u}\) is physical variable,
- \(k = \frac{\omega}{c} = \frac{\omega}{c(x)}\) is wavenumber and
- \(\mathbf{g}\) is source function.

Discretization. Finite difference method:

\[
\frac{1}{h^2} \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ \end{bmatrix} \mathbf{u}_{i,j} = \mathbf{g}_{i,j}
\]

Linear system \(\mathbf{A}\mathbf{u} = \mathbf{g}\)

- Symmetric
- Complex valued
- Indefiniteness for sufficient large \(k\)

Iterative Schemes

Indefiniteness of the linear system \(\mathbf{A}\mathbf{u} = \mathbf{g}\) leaves less choice, still best are Krylov subspace methods, and GMRES, in particular.

Shifted Laplace Preconditioner:

\[-\Delta \mathbf{u} - (\beta_1 + i\beta_2)k^2 \mathbf{u}(x, y)\]  

- with same boundary conditions
- spectrum of preconditioned system is bounded within unit circle
- outlayers of spectrum rushes to zero as \(k\) increases

- Multigrid works well for SL preconditioner than Helmholtz, because of inclusion of imaginary shift

Deflation

\[P = I - AQ\]  

with \(Q = ZE^{-1}Z^T\)

\[E = Z^TAZ\]

\(Z\) is deflation matrix.

In Multigrid Matrix, \(Z = R\), Restriction operator.

Numerical Experiments

<table>
<thead>
<tr>
<th>(k)</th>
<th>SLP</th>
<th>SLP with deflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>9</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>11</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>13</td>
</tr>
<tr>
<td>50</td>
<td>51</td>
<td>15</td>
</tr>
</tbody>
</table>

- Also \(h\)-independent scheme.

Conclusive remarks and further work

- sparse deflation (Multigrid deflation)
- \(h\)-independent solution

Further:

- Analysis of scheme (Local Fourier Analysis, in process)
- Proceeding to Wedge problem.