

An Scalable Helmholtz Solver Combining the Shifted Laplace Preconditioner With Multigrid Deflation

15th COPPER MOUNTAIN CONFERENCE ON MULTIGRID METHODS

27 March - 01 April 2011

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Outline

- Introduction
- Preconditioning
- Second-level preconditioning (*Deflation*)
- Fourier Analysis of two-level method
- Numerical experiments
- Conclusions

Introduction

Applications:

Acoustics,
Seismic waves,
Optics (Light waves) and
Electromagnetic

Our object:

To develop an iterative efficient iterative scheme to get acceptable numerical solution of the Helmholtz equation

The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x, y) - k^2(x, y) \mathbf{u}(x, y) = \mathbf{g}(x, y) \quad \text{in } \Omega$$

$\mathbf{u}(x, y)$ is the pressure field,

$k(x, y)$ is the wave number,

$\mathbf{g}(x, y)$ is the point source function and

Ω is domain bounded by Absorbing boundary conditions

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is normal direction to respective boundary.

Problem description

- Second order Finite difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4 - k^2 h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$: properties
 - Sparse & complex valued
 - Symmetric & Indefinite for large k
- Is traditionally solved by Krylov subspace method, they exploit the sparsity.

Preconditioning

- ILU and variants
- From Laplace to (complex) shifted Laplace preconditioner
- Shifted Laplace preconditioner (SLP)

$$M := -\Delta \mathbf{u} - (\beta_1 - \iota \beta_2) k^2 \mathbf{u}$$

- Results shows: $(\beta_1, \beta_2) = (1, 0.5)$ is shift of choice
- What does **SLP** do ? ?

Shifted Laplace Preconditioner

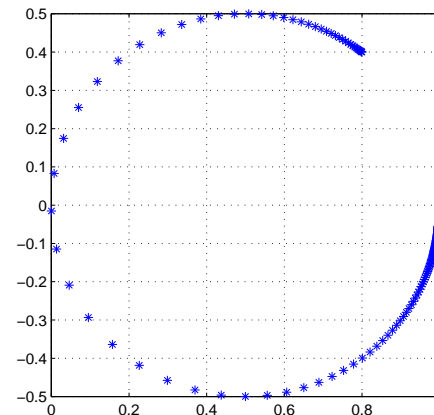
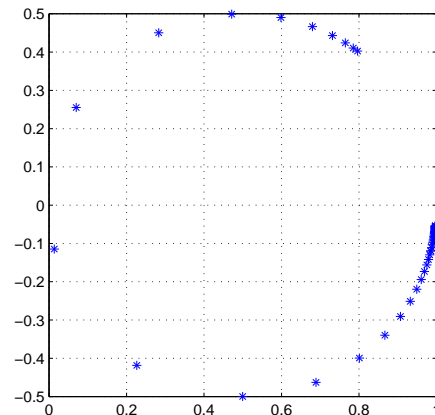
- Introduces damping, Multigrid approximation
- Norm of spectrum of preconditioned operator bounded above by 1
- Spectrum goes near to zero, as k increases.

Spectrum of $M^{-1}(1, 0.5)A$ for

$k = 30$

and

$k = 120$



Numerical results

Number of GMRES iterations. Shifts in preconditioner are (1, 0.5)

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	10	17	28	44	70	14
$n = 48$	10	17	28	38	49	308
$n = 64$	10	17	28	36	45	163
$n = 80$	10	17	27	35	44	116
$n = 160$	10	17	27	35	43	82
$n = 320$	10	17	27	35	42	80

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$n = 32$	5/10	8/17	14/28	26/44	42/70	13/14
$n = 48$	4/10	6/17	10/28	16/38	26/49	273/308
$n = 64$	4/10	6/17	8/28	12/36	18/45	173/163
$n = 80$	4/10	5/17	7/27	10/35	14/44	156/116
$n = 160$	3/10	4/17	5/27	6/35	8/43	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	10/80

with / without deflation.

Deflation

For any deflation subspace matrix

$$Z \in R^{n \times r}, \text{ with deflation vectors } Z = [z_1, \dots, z_r], \text{ rank } Z = r$$

$$P = I - AQ, \text{ with } Q = ZE^{-1}Z^T \text{ and } E = Z^T AZ$$

Solve $PAu = Pg$ preconditioned by M^{-1} or $M^{-1}PA = M^{-1}Pg$

If

$$\text{spec}(A) = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$$

and Z , the r eigenvectors then

$$\text{spec}(PA) = \{0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n\}$$

We use multigrid inter-grid transfer operator (Prolongation) as deflation matrix.

Multigrid Deflation

Setting $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P = I - AQ, \quad \text{with} \quad Q = I_h^{2h} E^{-1} I_{2h}^h \quad \text{and} \quad E = I_{2h}^h A_h^{2h}$$

where

P can be read as coarse grid correction and

Q the coarse grid operator

Fourier Analysis

Dirichlet boundary conditions for analysis.

With above deflation,

$$\text{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h)$$

is a complex valued function.

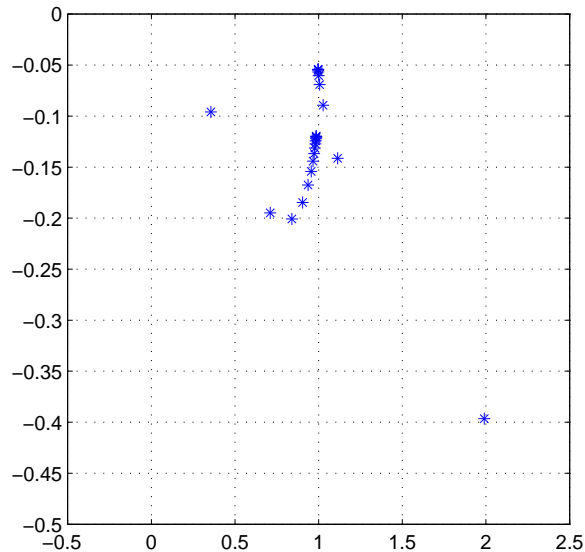
Setting $kh = 0.625$,

- Spectrum of $(I - PM^{-1}A)$ with shifts $(1, 0.5)$ near zero is wrapped and clustered around 1 with few outliers.
- Spectrum remains almost same, when imaginary shift is varied from 0.5 to 1.

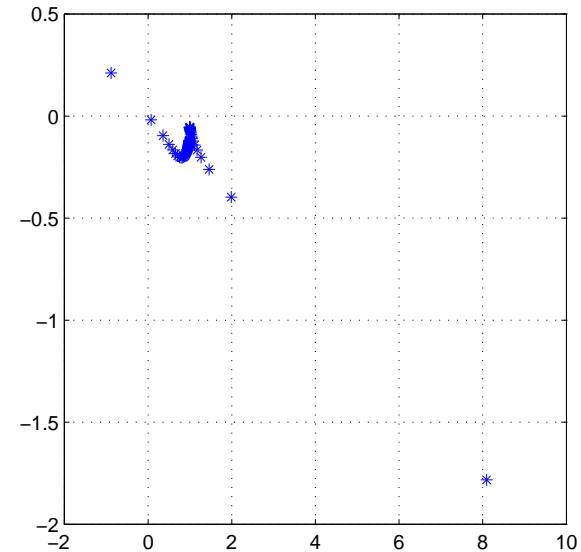
Fourier Analysis

Analysis gives spectrum clustered around one with few outliers.

$$k = 30$$



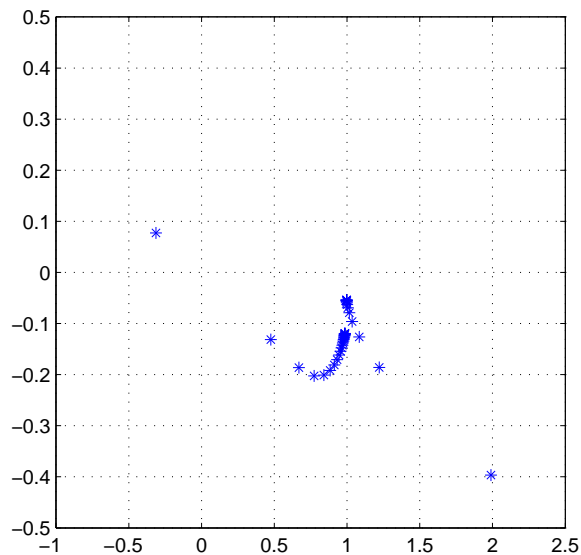
$$k = 120$$



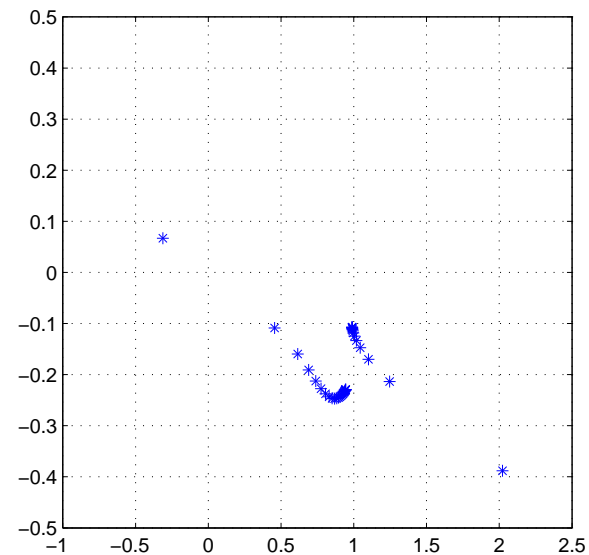
Fourier Analysis

Analysis tells increase in imaginary shift does not change spectrum.

$$(\beta_1, \beta_2) = (1, 0.5)$$



$$(\beta_1, \beta_2) = (1, 1)$$



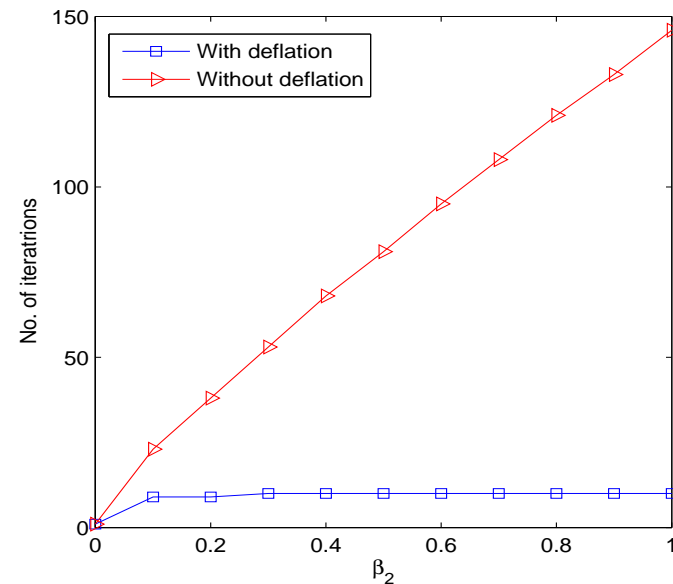
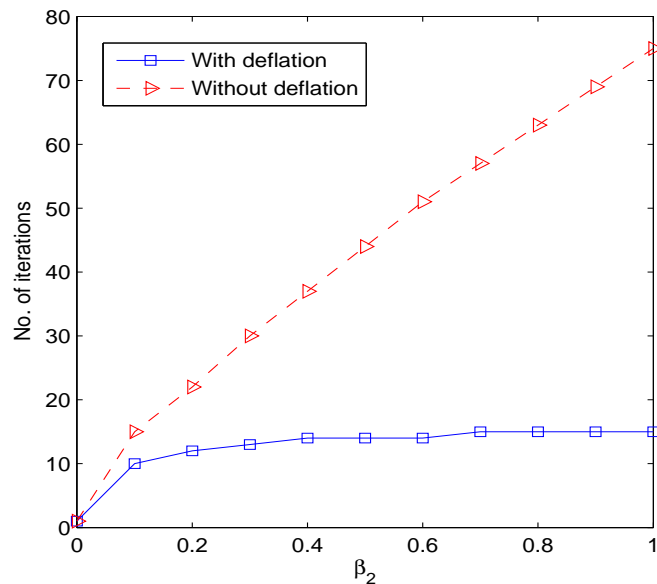
Numerical results

Sommerfeld boundary conditions are used for test problem.

Increase in imaginary shift in SLP ??

Constant wavenumber problem

Wedge problem



Numerical results

Number of GMRES iterations with/without deflation. Shifts in preconditioner are (1, 0.5)

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	5/10	8/17	14/28	26/44	42/70	13/14
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Numerical results

Number of GMRES iterations with/without deflation to solve a Wedge problem. Shifts in preconditioner are $(1, 0.5)$

Grid	<i>freq</i> = 10	<i>freq</i> = 20	<i>freq</i> = 30	<i>freq</i> = 40	<i>freq</i> = 50
74 × 124	7/33	20/60	79/95	267/156	490/292
148 × 248	5/33	9/57	17/83	42/112	105/144
232 × 386	5/33	7/57	10/81	25/108	18/129
300 × 500	4/33	6/57	8/81	12/105	18/129
374 × 624	4/33	5/57	7/80	9/104	13/128

Conclusions

- Parameter independent scheme
- Numerical results confirms analysis.
- Flexibility to increase imaginary shift, when deflation is combined with SLP.
- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- **Further** Multilevel scheme, solving the coarse grid system recursively.

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Thank You for Your Attention