A New Model Reduction Algorithm for Sequentially Semiseparable Matrices and Its Applications in System Identification

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27th June, 2014, Strasbourg, France

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Outline

Control and Identification of Distributed Systems

Control of spatially interconnected system arise in many applications

- multi-agent systems
- distributed power control
- control of PDEs
- ......

Challenge for control and identification of such systems

- High Computational Complexity

  N interconnected systems, each system with n states, $\mathcal{O}(n^3 N^3)$
- Memory Consumption

  $\mathcal{O}(n^2 N^2)$, not economical for large N.

Ways Out

Overcome the computational complexity

- Linear matrix inequality (LMI) approach

  Spatially invariant system transformed by fractional transformation, $\mathcal{O}(n^{2\alpha}N^{\alpha})$, where $2.5 < \alpha < 3.5$. [D'Andrea & Dullerud, TAC 2003.]

- Structured matrix approach

  Exploit the matrix structure, $\mathcal{O}(n^3N)$. [Rice & Verhaegen, TAC 2010, TAC 2011; Torres & van Wingerden, TAC 2014.]

- Approximate sparse inverse

  Exploit the sparsity of the system matrix, compute an approximate sparse inverse. [Haber & Verhaegen, TAC 2013; Lin & Jovanovic, TAC 2014.]

Reduce memory consumption

- Structured matrix approach

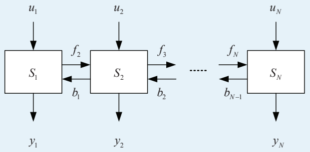
  $\mathcal{O}(n^2N)$ for sequentially semiseparable (SSS) matrix, $n \ll N$.

- Approximate sparse inverse

  Depends on the prescribed sparsity patten and approximation error, and $< \mathcal{O}(n^2N^2)$.

1-D Distributed Systems

Spatially Interconnected Systems



Local system

$$S_i: \begin{bmatrix} \dot{x}_i \\ f_{i+1} \\ b_{i-1} \\ y_i \end{bmatrix} = \begin{bmatrix} A_i & B_i^f & B_i^b & B_i \\ C_i^f & R_i & 0 & Q_i \\ C_i^b & 0 & W_i & V_i \\ C_i & G_i^f & G_i^b & D_i \end{bmatrix} \begin{bmatrix} x_i \\ f_i \\ b_i \\ u_i \end{bmatrix}$$

Global System Model

$$\bar{x} = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T$$

$$\bar{u} = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_N^T \end{bmatrix}^T$$

$$\bar{y} = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_N^T \end{bmatrix}^T$$

Global system

$$\begin{bmatrix} \dot{\bar{x}} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$$

where,

$$\bar{A} = \begin{bmatrix} A_1 & B_1^b C_2^b & B_1^b W_2 C_3^b & B_1^b W_2 W_3 C_4^b \\ B_2^f C_1^f & A_2 & B_2^b C_3^b & B_2 W_3 C_4^b \\ B_3^f R_2 C_1^f & B_3^f C_2^f & A_3 & B_3 C_4^b \\ B_4^f R_3 R_2 C_1^f & B_4^f R_3 C_2^f & B_4^f C_3^f & A_4 \end{bmatrix}$$

SSS Matrices and Properties

Definition

Let A be an $N \times N$ block matrix with SSS structure, then A can be written in the following block partitioned form

$$A_{i,j} = \begin{cases} U_i W_{i+1} \cdots W_{j-1} V_j^T, & \text{if } i < j; \\ D_i, & \text{if } i = j; \\ P_i R_{i-1} \cdots R_{j+1} Q_j^T, & \text{if } i > j. \end{cases}$$

and denoted as $A = \mathcal{SSS}(P_S,\ R_S,\ Q_s,\ D_s,\ U_s,\ W_s,\ V_s)$.

Properties of SSS Matrices

- $+$, $\times$, and $()^{-1}$ are closed under such structure.
- Complexity of at most $\mathcal{O}(r^3 N)$
- $+$, $\times$ will lead to growth of the rank of the off-diagonal blocks, r_i, where $r = \max_i r_i$.
- Model order reduction (MOR) is necessary to keep the computational complexity low.

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Model Reduction for SSS Matrices

Why Model Reduction?

The off-diagonal rank r defined by

$$r = max\left\{r_l, r_u\right\}$$

where

$$r_l = \max\left\{\text{size}(R_i)\right\}, \ r_u = \max\left\{\text{size}(W_i)\right\}$$

Basic matrix-matrix operations increase r (keep in mind $\mathcal{O}(r^3N)$).

What is Model Reduction?

To approximate the SSS matrix

$$A = \mathcal{SSS}(P_S,\ R_S,\ Q_s,\ D_s,\ U_s,\ W_s,\ V_s)$$

with

$$\tilde{A} = \mathcal{SSS}(\tilde{P}_S,\ \tilde{R}_S,\ \tilde{Q}_s,\ D_s,\ \tilde{U}_s,\ \tilde{W}_s,\ \tilde{V}_s)$$

where $\tilde{P}_S,\ \tilde{R}_S,\ \tilde{Q}_s,\ \tilde{U}_s,\ \tilde{W}_s,\ \tilde{V}_s$ have smaller size, and

$$\|A - \tilde{A}\| \leq \epsilon$$

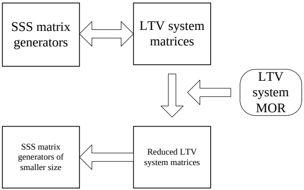
SSS Matrices and Its LTV System Realization

Motivation

Mixed-causal linear time-varying (LTV) system over finite time interval $[k_0,\ k_f]$,

$$\left.\begin{aligned}
\begin{bmatrix} x^c_{i+1} \\ x^a_{i-1} \end{bmatrix} &= \begin{bmatrix} R_i & \\ & W_i \end{bmatrix} \begin{bmatrix} x^c_i \\ x^a_i \end{bmatrix} + \begin{bmatrix} Q_i \\ V_i \end{bmatrix} u_i \\
y_i &= \begin{bmatrix} P_i & U_i \end{bmatrix} \begin{bmatrix} x^c_i \\ x^a_i \end{bmatrix} + D_i u_i \\
\bar{u} &= \begin{bmatrix} u_1^T, & u_2^T, & \dots & u_N^T \end{bmatrix}^T \\
\bar{y} &= \begin{bmatrix} y_1^T, & y_2^T, & \dots & y_N^T \end{bmatrix}^T
\end{aligned}\right\} \xRightarrow{N=4} \bar{y} = \begin{bmatrix} D_i & U_1 V_2 & U_1 W_2 V_3 & U_1 W_2 W_3 V_4 \\ P_2 Q_1 & D_2 & U_2 V_3 & U_2 W_3 V_4 \\ P_3 R_2 Q_1 & P_3 Q_2 & D_3 & U_3 V_4 \\ P_4 R_3 R_2 Q_1 & P_4 R_3 Q_2 & P_4 Q_3 & D_4 \end{bmatrix} \bar{u}$$

How to do Model Reduction?



Approximate Balanced Truncation for LTV Systems

Low Rank Approximation of the Gramians

Causal part of the mixed-causal LTV system,

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k \\ y_k = C_k x_k \end{cases}$$

over $[k_o, k_f]$, where $A_k \in \mathbb{R}^{M_{k+1} \times M_k}$, $B_k \in \mathbb{R}^{M_{k+1} \times m_k}$, and $C_k \in \mathbb{R}^{n_k \times M_k}$.

Controllability gramian $\mathcal{G}_c(k)$ and observability gramian $\mathcal{G}_o(k)$

$$\begin{aligned} \mathcal{G}_c(k+1) &= A_k \mathcal{G}_c(k) A_k^T + B_k B_k^T \\ \mathcal{G}_o(k) &= A_k^T \mathcal{G}_o(k+1) A_k + C_k^T C_k \end{aligned}$$

with $\mathcal{G}_c(k_o) = 0$ and $\mathcal{G}_o(k_f + 1) = 0$. $\mathcal{G}_c(k)$ and $\mathcal{G}_o(k)$ are symmetric positive semi-definite, usually of low numerical rank.

$$\begin{aligned} & \mathcal{G}_c(k) = L_k L_k^T, \quad L_k \in \mathbb{R}^{M_k \times r_k}, \quad r_k \le M_k, \\ \xrightarrow{\text{Low rank approx.}} \quad & \mathcal{G}_c(k) \approx \tilde{L}_k \tilde{L}_k^T, \quad \tilde{L}_k \in \mathbb{R}^{M_k \times r}, \quad r \le r_k. \end{aligned}$$

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Approximate Balanced Truncation for LTV Systems (cont'd)

Low Rank Update

Approximate $\mathcal{G}_c(k) \approx \tilde{\mathcal{G}}_c(k) = \tilde{L}_k \tilde{L}_k^T$, update $\tilde{\mathcal{G}}_c(k+1)$

- $\left[\begin{array}{c|c} A_k \tilde{L}_k & B_k \end{array}\right] = U \Sigma V^T$
- $U = \left[\begin{array}{c|c} U_1 & U_2 \end{array}\right]$, $\Sigma = \left[\begin{array}{c|c} \Sigma_1 & \\ \hline & \Sigma_2 \end{array}\right]$
- $\tilde{L}_{k+1} = U_1 \Sigma_1$

Approximate Balanced Truncation

Balancing via

$$\tilde{\mathcal{G}}_c^T(k) \tilde{\mathcal{G}}_o(k) = U_k \Sigma_k V_k^T$$

truncation by

$$\Pi_l(k) = \tilde{\mathcal{G}}_o(k) V_k \Sigma_k^{-\frac{1}{2}}, \ \Pi_r(k) = \tilde{\mathcal{G}}_c(k) U_k \Sigma_k^{-\frac{1}{2}}$$

gives

$$\begin{cases} \tilde{x}_{k+1} = \Pi_l(k+1) A_k \Pi_r(k) \tilde{x}_k + \Pi_l(k+1) B_k u_k \\ y_k = C_k \Pi_r(k) \tilde{x}_k \end{cases}.$$

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Extended Kalman Filter for 1-D System Identification

1-D Heat Equation

Heat equation over $x \in [0, 1]$

$$\frac{\partial u(x,t)}{\partial t} = \nabla \cdot (k(x)\nabla u(x,t)) - \frac{1}{2}u(x,t)$$

$u(x, t_0) = u_0$ on $\partial\Gamma$, $\nabla\cdot$ divergence operator and ∇ gradient operator, $k(x)$ the heat conduction coefficients.

$$\text{PDE} \overset{\text{FEM/FDM}}{\Longrightarrow} \text{ODE} \overset{\text{Euler Method}}{\Longrightarrow} \begin{cases} T_{k+1} = A(\theta)T_k + B(\theta)u_k \\ y_k = C(\theta)T_k \end{cases}$$

with $\theta = \{k(x), \Delta t\}$

$$\begin{cases} T_{k+1} = A(\theta)T_k + B(\theta)u_k + \omega_k \\ y_k = C(\theta)T_k + v_k \end{cases}$$

where $E(\omega_k\omega_j^T) = Q\delta_{kj}$, $E(v_k v_j^T) = R\delta_{jk}$, $E(\omega_k v_j^T) = S\delta_{kj}$, $E(x_0) = 0$, $E(x_0 x_0^T) = \Pi_x$ and $E(\theta\theta^T) = \Pi_\theta$.

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Extended Kalman Filter for SysID

Innovations Form

$$\begin{cases} \widehat{T}_{k+1} = A(\widehat{\theta}_k)\widehat{T}_k + B(\widehat{\theta}_k)u_k + K_k\left(y_k - C(\widehat{\theta}_k)\widehat{T}_k\right) \\ \widehat{\theta}_{k+1} = \widehat{\theta}_k + L_k\left(y_k - C(\widehat{\theta}_k)\widehat{T}_k\right) \end{cases}$$

Kalman gains

$$\begin{bmatrix} K_k \\ L_k \end{bmatrix} = \left(\begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix} \Xi_k \begin{bmatrix} C_k & N_k \end{bmatrix}^T + \begin{bmatrix} S \\ 0 \end{bmatrix} \right) P_k^{-1}.$$

and $A_k = A(\widehat{\theta}_k)$, $M_k = \frac{\partial}{\partial \theta}\left(A(\theta)\widehat{T}_k + B(\theta)u_k\right)|_{\theta=\widehat{\theta}_k}$, $C_k = C(\widehat{\theta}_k)$,
$N_k = \frac{\partial}{\partial \theta}\left(C(\theta)\widehat{T}_k\right)|_{\theta=\widehat{\theta}_k}$ where

$$P_k = \begin{bmatrix} C_k & N_k \end{bmatrix} \Xi_k \begin{bmatrix} C_k & N_k \end{bmatrix}^T + R$$

$$\Xi_{k+1} = \begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix} \Xi_k \begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix}^T - \begin{bmatrix} K_k \\ L_k \end{bmatrix} P_k \begin{bmatrix} K_k \\ L_k \end{bmatrix}^T + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

Numerical Experiments

Choose $k(x) \in 1 + \frac{2}{5}\mathcal{U}[-1,1]$, $\omega \sim \mathcal{N}(0, 10^{-6})$, $v \sim \mathcal{N}(0, 10^{-6})$

Set initial guess $\Xi_x = \Xi_\theta = I$, $k_0 = \mathbf{1}$. Stop criteria $\frac{\|\hat{\theta}_k - \theta^\star\|_2}{\|\theta^\star\|_2} \leq 5 \times 10^{-4}$

Random run the numerical experiments 5 times.

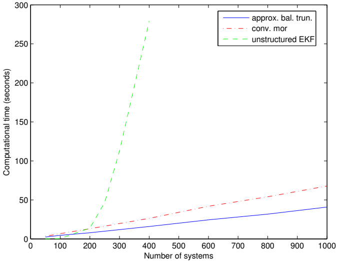

Figure: Average computational time

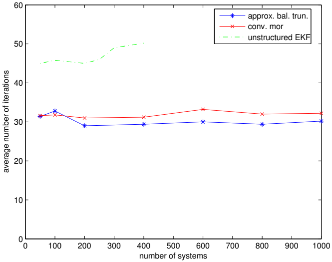

Figure: Average number of iterations

Numerical Experiments (cont'd)

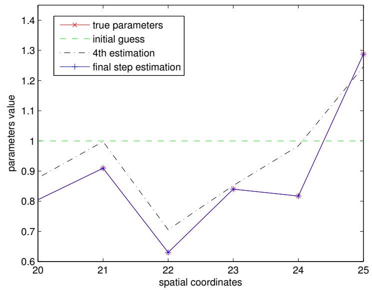

Figure: Estimated parameters

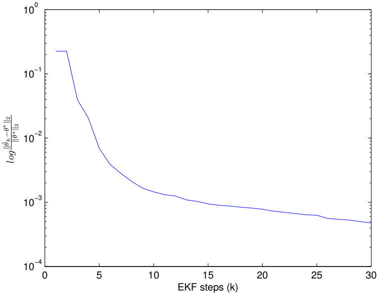

Figure: Asypototic convergence

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Conclusions & Remarks

- SSS matrix computations yield linear computational complexity for EKF for 1-D SysID.
- The approximate balanced truncation is computationally cheaper than the conventional method.
- Both MOR algorithms give linear computational complexity.
- To extend to higher dimensional systems, such as 2-D or 3-D, multilevel SSS matrix is necessary.
- For multilevel SSS matrices, structure preserving MOR is the key, still an open problem.

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