An Scalable Helmholtz Solver Combining the Shifted Laplace Preconditioner With Multigrid Deflation

Sparse Days

September 6-7th, 2011

CERFACS, Toulouse, France.

A.H.Sheikh, D.Lahaye, C.Vuik

Delft Institute of Applied Mathematics

TU Delft

Delft University of Technology

Title of the slide

- Introduction
- Preconditioning
- Second-level preconditioning (`Deflation`)
- Fourier Analysis of two-level method
- Numerical experiments
- Conclusions

Introduction

Applications:

Acoustics,

Seismic waves,

Optics (Light waves) and

Electromagnetic

Our object:

To develop an iterative efficient iterative scheme to get acceptable numerical solution of the Helmholtz equation

The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta\mathbf{u}(x,y) - k^2(x,y)\mathbf{u}(x,y) = \mathbf{g}(x,y) \text{ in } \Omega$$

$\mathbf{u}(x,y)$ is the pressure field,

$\mathbf{k}(x,y)$ is the wave number,

$\mathbf{g}(x,y)$ is the point source function and

Ω is domain bounded by Absorbing boundary conditions

$$\frac{\partial\mathbf{u}}{\partial n} - \iota\mathbf{u} = 0$$

n is normal direction to respective boundary.

Problem description

- Second order Finite difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4 - k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $Au = g$: properties

  Sparse & complex valued

  Symmetric & Indefinite for large k

- Is traditionally solved by Krylov subspace method, they exploit the sparsity.

Preconditioning

- ILU and variants
- From Laplace to complex shifted Laplace preconditioner (2005)
- Shifted Laplace preconditioner (SLP)

$$M := -\Delta \mathbf{u} - (\beta_1 - \iota\beta_2)k^2\mathbf{u}$$

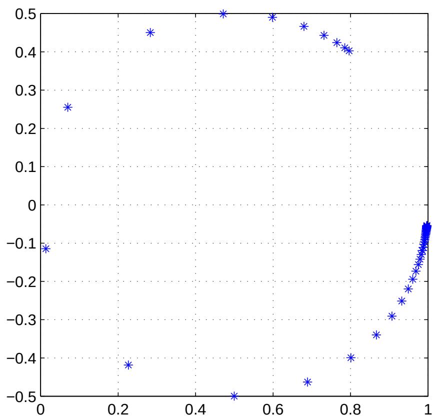
- Results shows: $(\beta_1, \beta_2) = (1, 0.5)$ is shift of choice
- What does SLP do ? ?

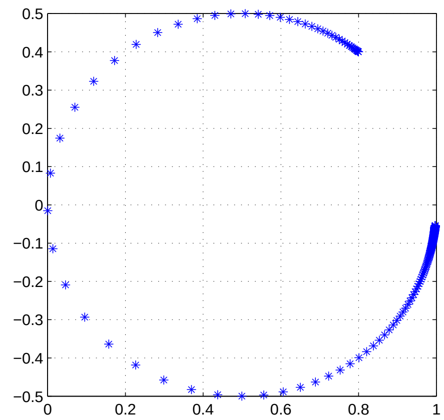
Shifted Laplace Preconditioner

- Introduces damping, Multigrid approximation
- Norm of spectrum of preconditioned operator bounded above by 1
- Spectrum goes near to zero, as k increases.

Spectrum of $M^{-1}(1, 0.5)A$ for

$k = 30$ and $k = 120$





Numerical results

Number of GMRES iterations. Shifts in preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**10**	17	28	44	70	14
$n = 48$	10	**17**	28	38	49	308
$n = 64$	10	17	**28**	36	45	163
$n = 80$	10	17	27	**35**	44	116
$n = 160$	10	17	27	35	**43**	82
$n = 320$	10	17	27	35	42	**80**

Numerical results

Number of GMRES iterations. Shifts in preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 48$	4/10	**6/17**	10/28	16/38	26/49	273/308
$n = 64$	4/10	6/17	**8/28**	12/36	18/45	173/163
$n = 80$	4/10	5/17	7/27	**10/35**	14/44	156/116
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

with / without deflation.

Deflation: definition

For any deflation subspace matrix

$$Z \in R^{n\times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank Z = r$$

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \ \text{and} E = Z^T AZ$$

Solve $PAu = Pg$ preconditioned by M^{-1} or $M^{-1}PA = M^{-1}Pg$

For e.g. , if

$$\textbf{spec}(A) = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$$

and Z, the r eigenvectors then

$$\textbf{spec}(PA) = \{0, ..., 0, \lambda_{r+1}, ...\lambda_n\}$$

In multigrid deflation, inter-grid transfer operator (Prolongation) as deflation matrix.

Deflation

Setting $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P = I - AQ, \quad \text{with} \quad Q = I_h^{2h} E^{-1} I_{2h}^h \;\text{ and }\; E = I_{2h}^h A_h^{2h}$$

where

P can be read as coarse grid correction and

Q the coarse grid operator (or Galerkin operator).

E^{-1} need to be computed, and this leads to a multilevel algorithm.

Fourier Analysis

1D Helmholtz model.

Typically, Dirichlet boundary conditions.

$$\textbf{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h, l)$$

is a complex valued function.

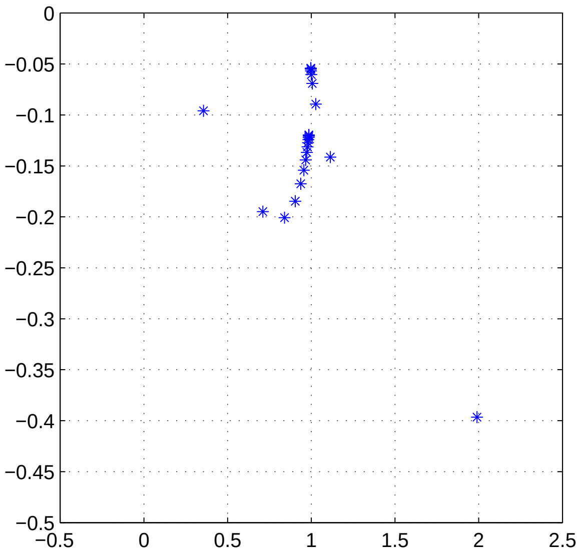
Setting $kh = 0.625$ and $(\beta_1, \beta_2) = (1, 0.5)$, we see

- Spectrum of $(I - PM^{-1}A)$ with shifts $(1, 0.5)$ near zero is wrapped and clustered around 1 with few outliers.
- Spectrum remains almost same, when imaginary shift is varied from 0.5 to 1.

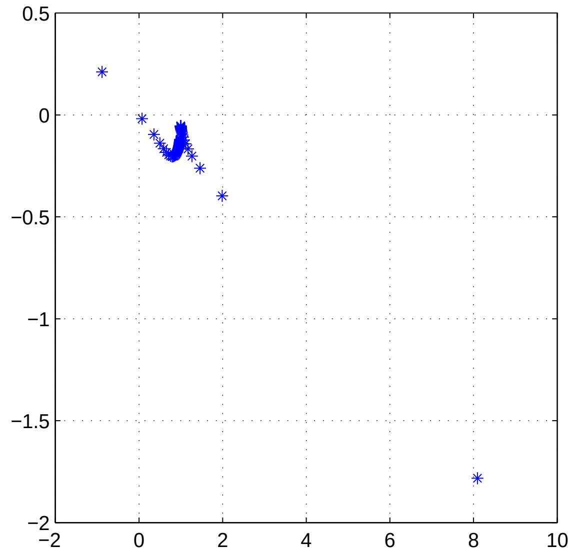
Fourier Analysis

Analysis shows, deflation pushes spectrum around one with few outliers.

$k = 30$



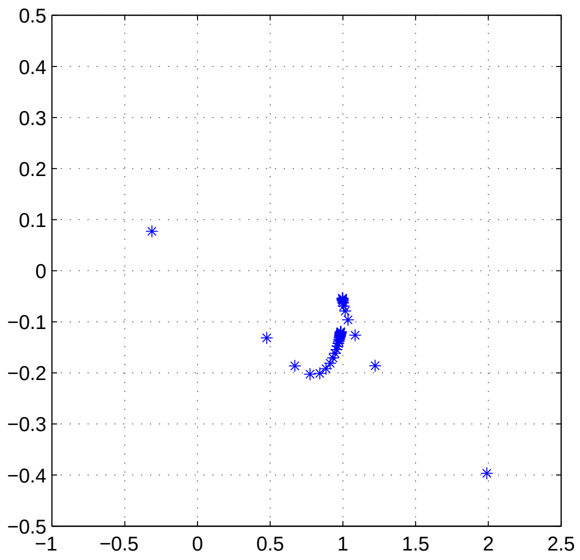
$k = 120$



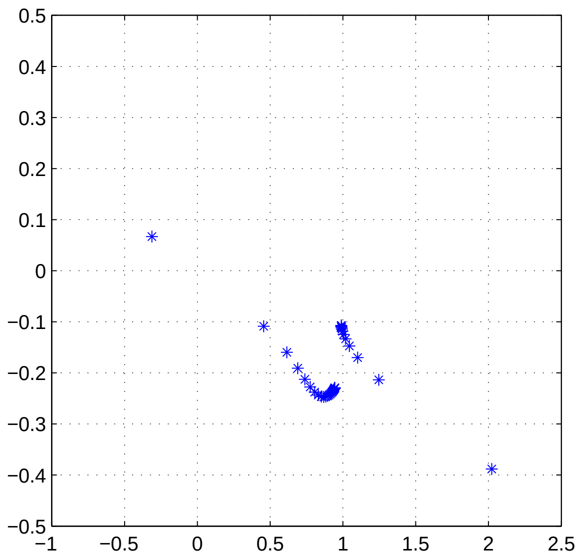
Fourier Analysis

Analysis tells increase in imaginary shift does not change spectrum.

$(\beta_1, \beta_2) = (1, 0.5)$



$(\beta_1, \beta_2) = (1, 1)$

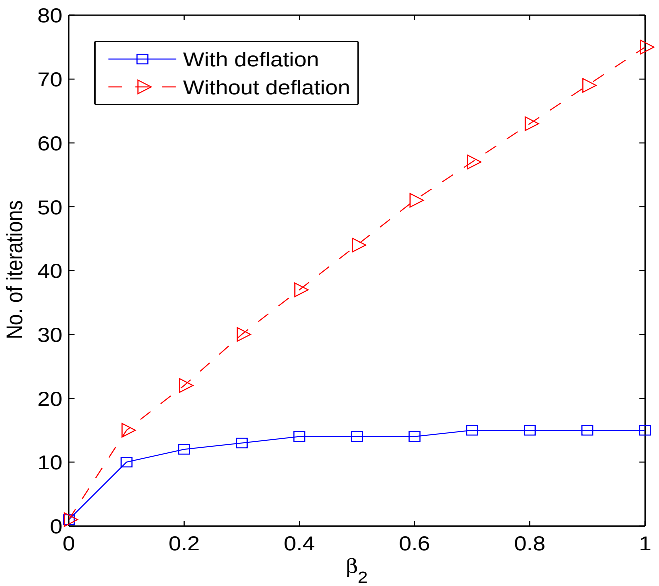


Numerical results

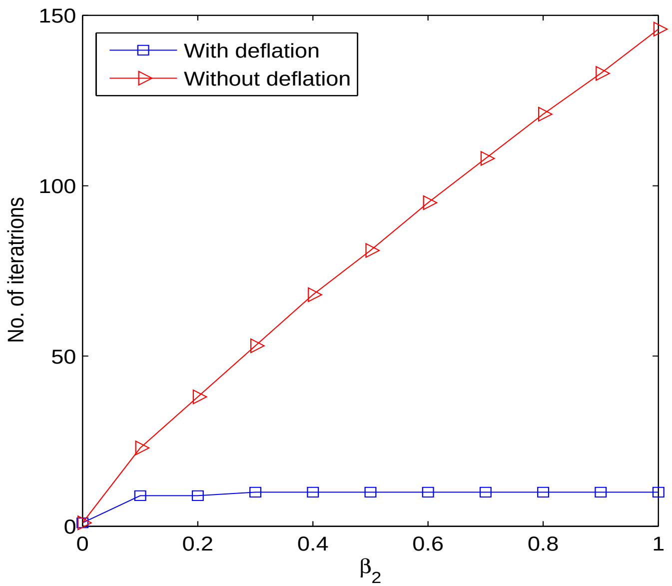
Sommerfeld boundary conditions are used for test problem.

Increase in imaginary shift in SLP ??

Constant wavenumber problem



Wedge problem



Numerical results

- Number of GMRES iterations with/without deflation. Shifts in preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 48$	4/10	**6/17**	10/28	16/38	26/49	273/308
$n = 64$	4/10	6/17	**8/28**	12/36	18/45	173/163
$n = 80$	4/10	5/17	7/27	**10/35**	14/44	156/116
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Numerical results

Number of GMRES iterations with/without deflation to solve a Wedge problem. Shifts in preconditioner are $(1, 0.5)$

Grid	$freq = 10$	$freq = 20$	$freq = 30$	$freq = 40$	$freq = 50$
74×124	**7/33**	20/60	79/95	267/156	490/292
148×248	5/33	**9/57**	17/83	42/112	105/144
232×386	5/33	7/57	**10/81**	25/108	18/129
300×500	4/33	6/57	8/81	**12/105**	18/129
374×624	4/33	5/57	7/80	9/104	**13/128**

Conclusions

- (Almost) Parameter independent scheme.
- 1D problem analyzed.
- Numerical results confirms analysis.
- Flexibility to increase imaginary shift, when deflation is combined with SLP.
- Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
- Further Multilevel scheme, recursively for coarse problem in deflation.
- Further LFA for 2D problem, taking into account multigrid solution of SL preconditioner.

References

- M. van Gijzen, Y. Erlangga, C. Vuik , Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian, SIAM J. Sc. Comp. 29 (2007) 1942-1958
- Y. Erlangga, R. Nabben, On a multilevel Krylov method for the Helmholtz equation preconditioned by shifted Laplacian, Electronic Transaction on Num. Analysis (ETNA) 31 (2008) 403-424.
- A.H. Sheikh, D. Lahaye, C. Vuik, A scalable Helmholtz solver combining the shifted Laplace perconditioner with multigrid deflations, DIAM Tech. Report 11-01, TU Delft, Netherlands
- J. Tang, S. MacLachlan, R. Nabben, C. Vuik, A comparison of two-level preconditioners based on multigrid and deflation, SIAM. J. Matrix Anal. and Appl. 31 (2010) 1715-1739.

Thank You for Your Attention