About Deflation Preconditioner for Helmholtz Problem

Group Talk Series

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Contents

- Helmholtz and SLP
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The Helmholtz equation

The Helmholtz equation without damping

$$-\Delta \mathbf{u}(x_i) - k^2(x_i)\mathbf{u}(x_i) = \mathbf{g}(x_i) \text{ in } \Omega$$

$\mathbf{k}(x_i)$ is the wave number,
$\mathbf{g}(x_i)$ is the point source function and
Ω is the domain. Absorbing boundary conditions are used on Γ.

$$\frac{\partial \mathbf{u}}{\partial n} - \iota \mathbf{u} = 0$$

n is the unit normal vector pointing outwards on the boundary.

Discretization

- Second order Finite Difference stencil:

$$\begin{bmatrix} & -1 & \\ -1 & 4-k^2h^2 & -1 \\ & -1 & \end{bmatrix}$$

- Linear system $A_h u_h = g_h$ holds properties:

  Sparse & complex valued

  Symmetric & Indefinite for large k

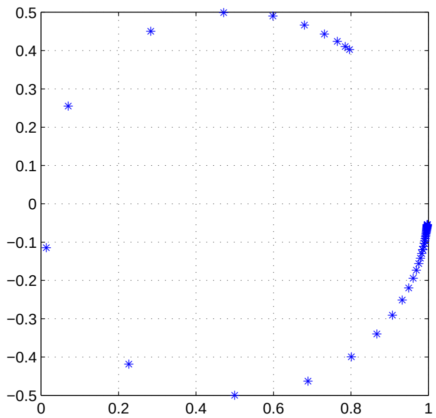
- For high resolution a very fine grid is required: $30-60$ gridpoints per wavelength (or $\approx 5-10\times k$) $\rightarrow A_h$ is extremely large!
- Traditionally solved by a Krylov subspace method, which exploits the sparsity.

Shifted Laplace Preconditioner

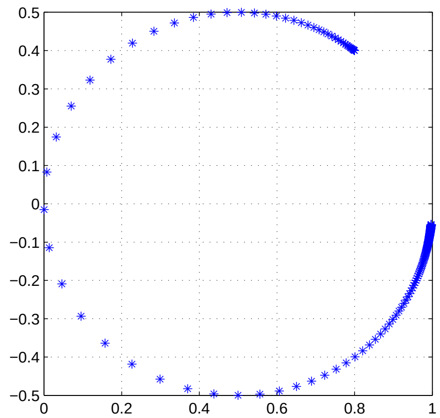
- $M(\beta_1, \beta_2) := -\Delta - (\beta_1 - \iota\beta_2)k^2 I$
- Spectrum encounters near-zero eigenvalues, as k increases.

Spectrum of $M^{-1}A$, where $(\beta_1, \beta_2) = (1, 0.5)$

$k = 30$



$k = 120$



Why Deflation!!

Number of GMRES iterations. Shifts in the preconditioner are $(1, 0.5)$

Grid	$k = 10$	$k = 20$	$k = 30$	$k = 40$	$k = 50$	$k = 100$
$n = 32$	**5/10**	8/17	14/28	26/44	42/70	13/14
$n = 64$	4/10	**6/17**	8/28	12/36	18/45	173/163
$n = 96$	3/10	5/17	**7/27**	9/35	12/43	36/97
$n = 128$	3/10	4/17	6/27	**7/35**	9/43	36/85
$n = 160$	3/10	4/17	5/27	6/35	**8/43**	25/82
$n = 320$	3/10	4/17	4/27	5/35	5/42	**10/80**

Diagonal entries where 20 gp/wl

with / without deflation

Deflation: or two-grid method

Deflation, a projection preconditioner

$$P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \ \text{ and } \ E = Z^T AZ$$

where,

$$Z \in R^{n \times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \ \ rank(Z) = r \leq n$$

Along with a traditional preconditioner M, deflated preconditioned system reads

$$PM^{-1}Au = PM^{-1}g.$$

The choice of deflation vectors: spectrum of matrix, physics of problem, etc

Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I_h^{2h}$ and $Z^T = I_{2h}^h$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I_h^{2h} A_{2h}^{-1} I_{2h}^h \text{ and } A_{2h} = I_{2h}^h A_h I_h^{2h}$$

where

P_h can be interpreted as a coarse grid correction and

Q_h as the coarse grid operator

Deflation: Implementation

Deflation can be implemented alongwith SLP M_h,

$$M_h^{-1} P_h A_h u_h = M_h^{-1} P_h g_h$$

$A_h u_h = g_h$ is preconditioned by two-level preconditoner $M_h^{-1} P_h$.

For large problems, A_{2h} is enough large to invert exactly and inversion of A_{2h} is sensitive, since P_h deflates spectrum to zero.

To do is: Solve A_{2h} iteratively to required accuracy on certain levels, and shift the deflated spectrum to λ_h^{max} by adding shift in two level preconditoner, we get **ADEF1** preconditioner

$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_h^{max} Q_h$$

Deflation: MLKM

MultiLevel Krylov Method [a], if $\hat{A}_h = M_h^{-1} A_h$, and develop $\hat{P}_h$ using $\hat{A}_h$ (instead A_h) will be

$$\hat{P}_h = I_h - \hat{A}_h \hat{Q}_h,$$

where

$$\hat{Q}_h = I_{2h}^h \hat{A}_{2h}^{-1} I_h^{2h} \text{ and } \hat{A}_{2h} = I_h^{2h} \hat{A}_h I_{2h}^h = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$$

Construction of coarse matrix A_{2h} at level $2h$ costs inversion of preconditioner at level h.

Approximate A_{2h} ?

Ideal	**Practical**
$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$	$A_{2h} = I_h^{2h} (M_h^{-1} A_h) I_{2h}^h$
	$A_{2h} \approx I_h^{2h} I_{2h}^h M_{2h}^{-1} A_{2h}$

[a] Erlangga, Y.A and Nabben R., ETNA 2008

ADEF1 v MLKM

Differentiating ADEF1 and MLKM, assuming $\lambda_{max} = 1$ and left preconditioning

ADEF1	**MLKM***
$P_{(ADEF1)} = M_h^{-1}(I_h - A_h Q_h) + Q_h$	$P_{(MLKM)} = I_h - \hat{A}_h \hat{Q}_h + \hat{Q}_h$
Applocation on $Au = g$	Application on $\hat{A}u = \hat{g}$

Cost comparison

Application cost per iteration at two levels

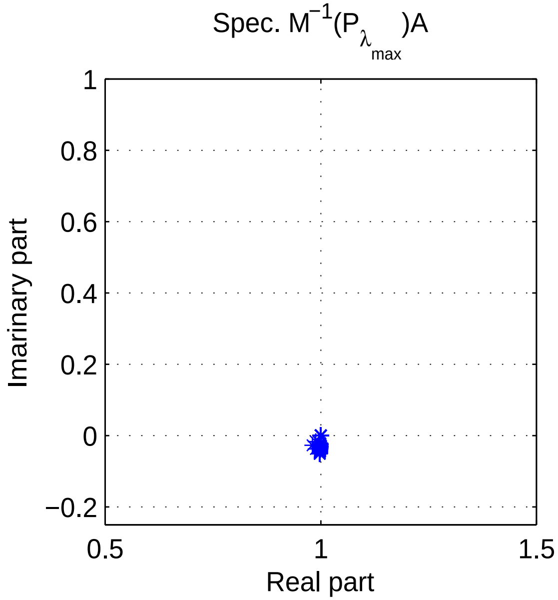
For some vector v,

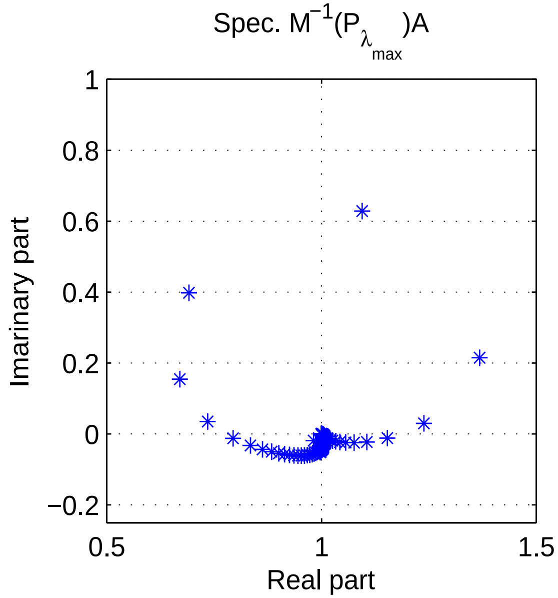
	ADEF1	MLMG
$A_h v$	1	1
$M_h^{-1} v$	1	2
$Q_h v$: $I_h^{2h} v$	1	1
$Q_h v$: $I_{2h}^{h} v$	1	1
$Q_h v$: M_{2h}^{-1}	0	1
$I_{2h}^{h} I_h^{2h} v$	0	1

Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM [b],
$k = 160$ and 20 gp/wl

Ideal | **Practical**



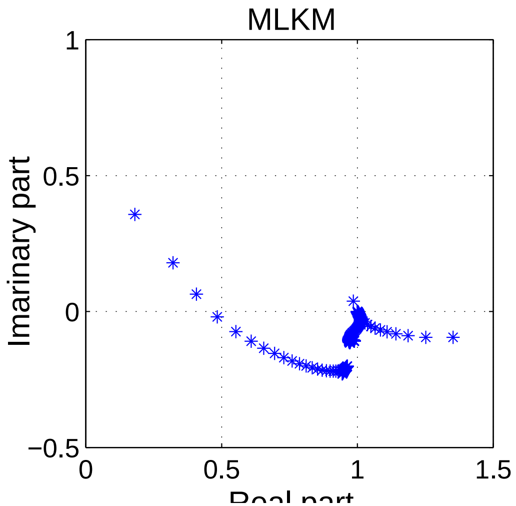


[b] Two-level

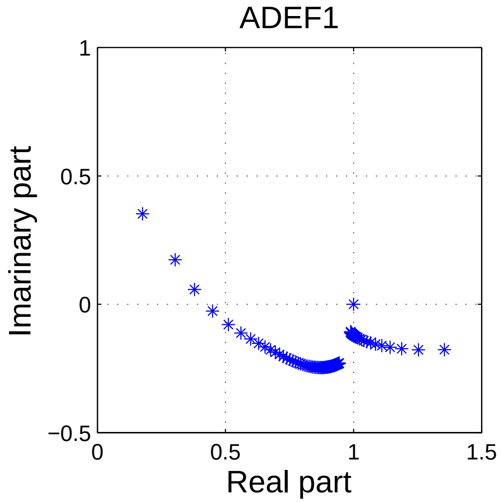
Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM and ADEF1;
$k = 160$ and 10 gp/wl

MLKM



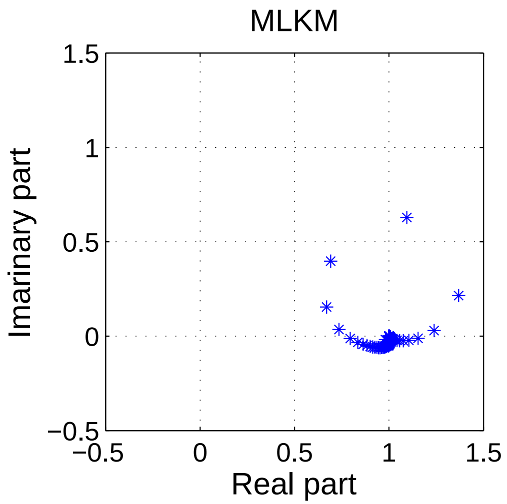
ADEF1



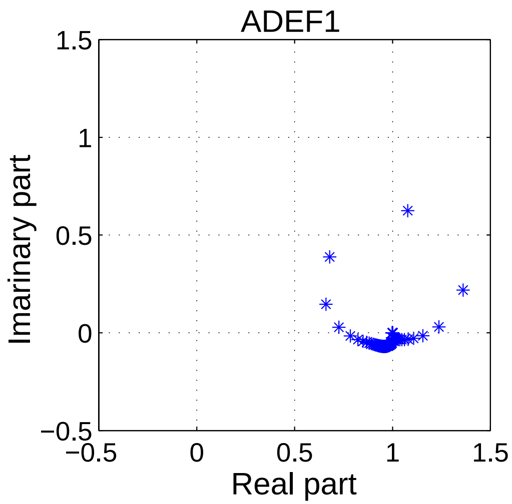
Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM and ADEF1;
$k = 160$ and 20 gp/wl

MLKM



ADEF1

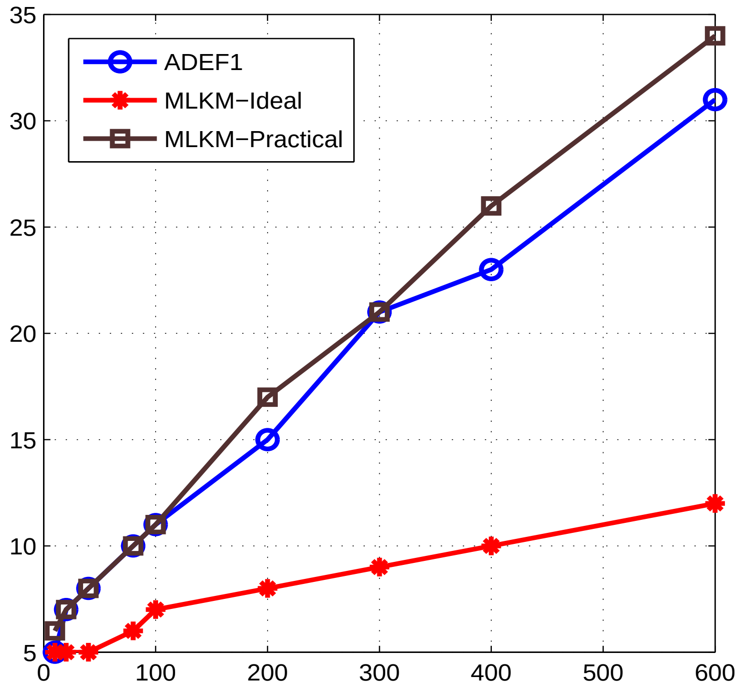


Numerical results

One Dimensional Helmholtz with Som. BCs.

Wave number against Krylov iterations

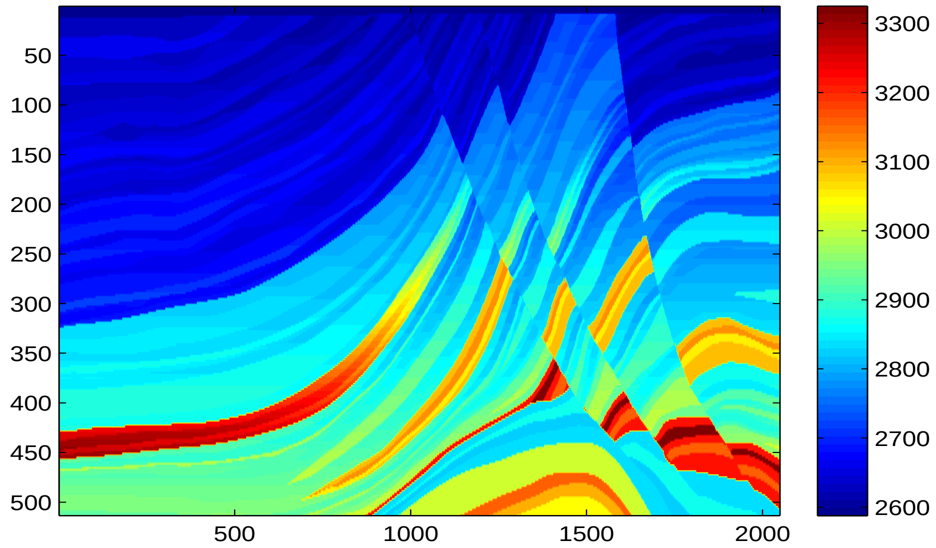
Two level solver



Comparison of number of iterations by ADEF1 and MLKM.

Adapted Marmousi Problem

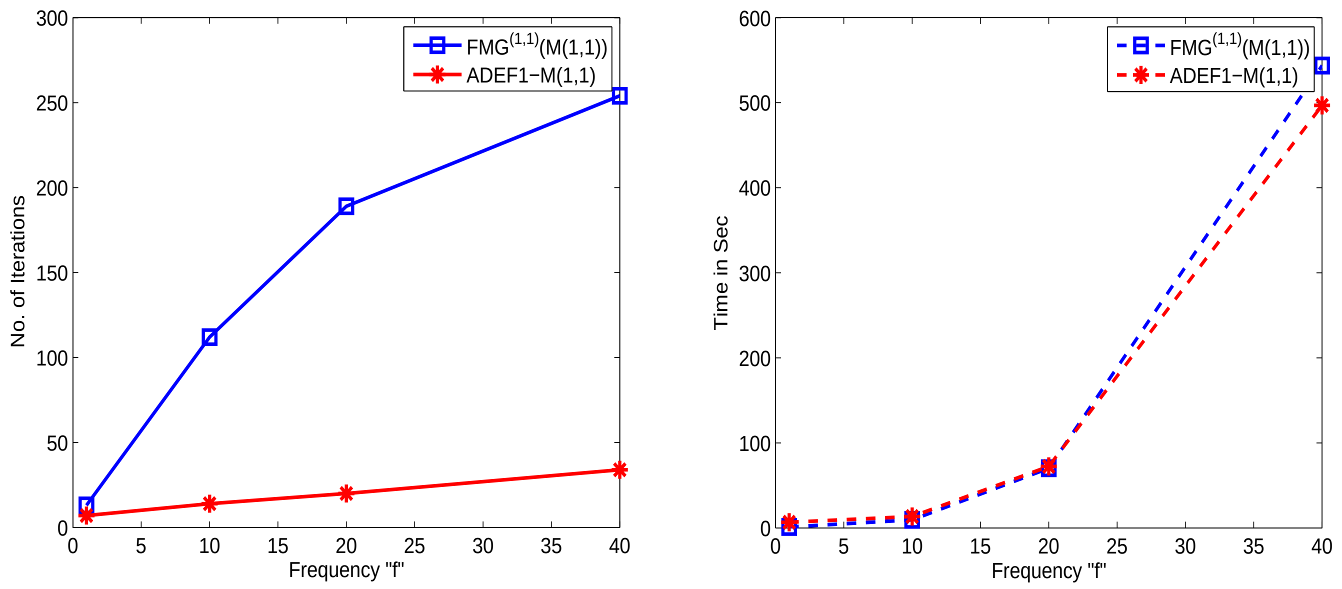
Reduced velocity contrast: $2587 \leq c(x, y) \leq 3325$



Numerical Results

Marmousi problem ADEF1 performance compared with SLP

No. of Iter. "t" in sec.



Numerical results

Three Dimensional Helmholtz on unit cube domain with sommerfeld boundary conditions on all faces.

	Wavenumber k							
Solver Type	5	10	15	20	30	40	60	80
SL Prec.	11	15	21	29	47	74	118	185
ADEF1-V(8,2,1)	11	15	21	28	44	67	101	153
ADEF1-F(8,2,1)	9	10	11	11	13	16	22	28

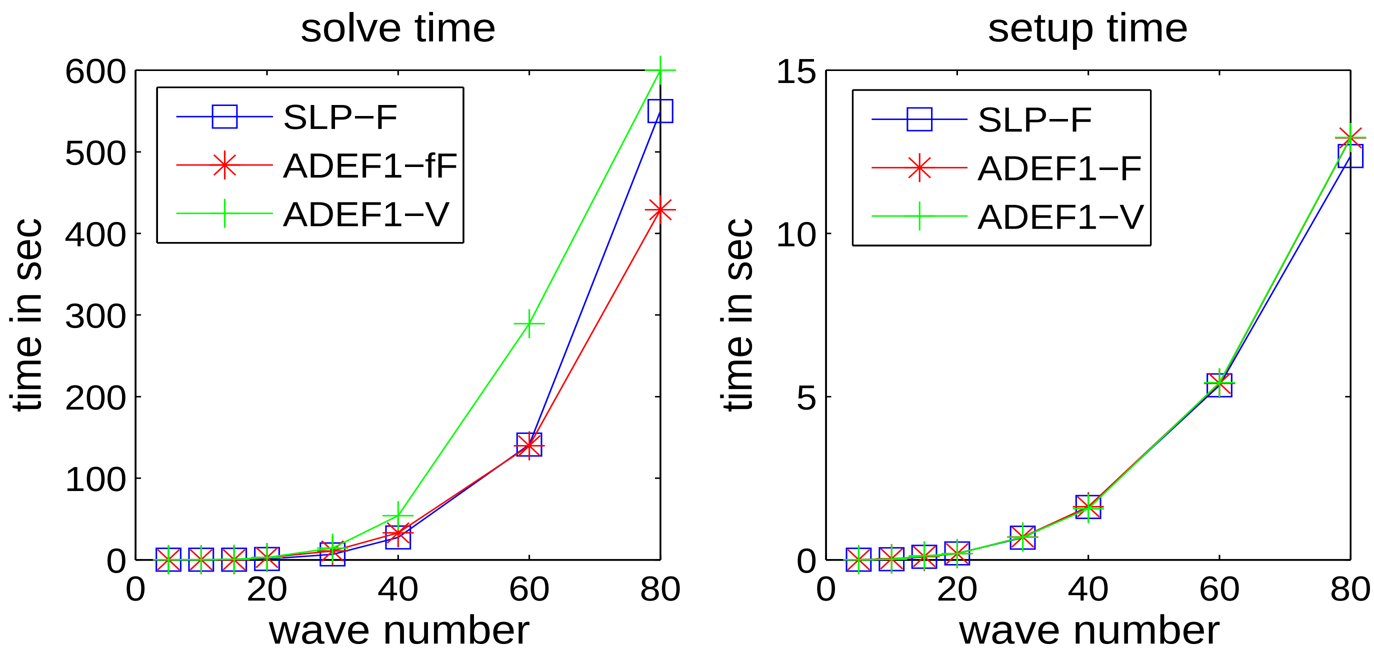
SL Prec. : Only shifted Laplace preconditioner

ADEF1-F : Multilevel solver , Fcycle for slp.

ADEF1-V : Multilevel solver , Vcycle for slp.

Numerical results

Petsc time for;



ADEF1 solve time and Setup time.

Numerical results

Algebraic deflation vectors ?

Regular mesh FEM discretization of 2D Helmholtz, Sommerfeld BCs.

Algebraic deflation vectors.

ADEF1 solver.

	Wavenumber k						
Solver	10	20	40	80	120	160	200
SL Prec.	22	43	72	128	178	232	278
2Lev	6	8	10	15	20	26	32
ADEF1-F	7	8	10	16	19	24	27

For 3D Helmholtz, some unexpected and unacceptable results were observed!

Conclusions

- Concern about solve time problem Marmousi: working on it.
- Damping improves ADEF1 performance, like SLP. Results not included.
- FEM discretization and algebraic deflation vectors in 3D are not favorable! Why, open question!
- Flexible in choosing larger imaginary shift.
- Further research 3D constrasted wavenumber problem, Different shifts in SLP at different levels, ...

References

- Y.A. Erlangga and R. Nabben. On a multilevel Krylov method for the Helmholtz equation preconditioned by shifted Laplacian. ETNA, 2008.
- M.B. van Gijzen, Y.A. Erlangga and C. Vuik. Spectral analysis of the discrete Helmholtz operator preconditioned with a shifted Laplacian. SIAM J.of Sc. Comp. 2007.
- J.M. Tang. Two level preconditioned Conjugate Gradient methods with applications to bubbly flow problems. PhD Thesis, DIAM TU Delft 2008.
- A.H. Sheikh, D. Lahaye and C. Vuik. On the convergence of shifted Laplace preconditioner combined with multi-grid deflation. NLAA 2013

Thank you!