Discrete Structures
(Discrete Mathematics)

Relations

Relations

- The connections between people and things.

- Between people, family relation
  - 'to be brothers' — x is a brother of y
  - 'to be older' — x is older than y
  - 'to be parents' — x and y are parents of z
- Between things, numerical relations
  - 'to be greater than' — $x < y$ on the real numbers
  - 'to be divisible by' — x is divisible by y on the set of integers
- Between things and people, legal relations
  - 'to be an owner' — x is an owner of y
  - 'to be std of class' — ***Trump*** is a std of class 24DS

Cartesian Product

- The Cartesian product of sets A and B, denoted by A × B, is the set of all ordered pairs of elements from A and B.
- $A \times B = \{(a, b) | a \in A \land b \in B\}$

- The elements of the Cartesian product are ordered pairs. In particular $(a, b) = (c, d) \leftrightarrow (a = c) \land (b = d)$

- $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$
- $\{Mon, Tue\} \times \{Jan, Feb\} = \{(Mon, Jan), (Mon, Feb), (Tue, Jan), (Tue, Feb)\}$

Cartesian Product of More Than Two Sets

- Instead of ordered pairs we may consider ordered triples, or, more general, k-tuples
- (a, b), an ordered pair
- (a, b, c), an ordered triple
- (a, b, c, d), an ordered quadruple
- $(a_1, a_2, \ldots, a_k)$, a k-tuple

- Pairs, triples, quadruples, and k-tuples are elements of Cartesian products of 3, 4, and k sets, respectively

Relations

- Relationship between elements of sets are represented using the structure called a relation, which is just a subset of the cartesian product of sets.

Definition:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be sets. A binary relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a subset of $\boldsymbol{A} \times \boldsymbol{B}$.

Binary Relation

- Let A and B be sets. A binary relation from set A to set B is any subset of $A \times B$.
- Binary relations represent relationship between two sets.

Example

- $A = \{Ali, Junaid, Maria, Hassan\}$
- $B = \{CSC102,\ CSC105,\ CSC222,\ CSC106\}$

  R: "$relation\ of\ students\ enrolled\ in\ courses$"

- $R = \{(Junaid, CSC102),\ (Maria, CSC222)\}$

- $(\text{Maria}, \text{CSC102}) \notin R$
- $(Junaid, CSC222) \notin R$
- $(Maria, CSC106) \notin R$

More Relations

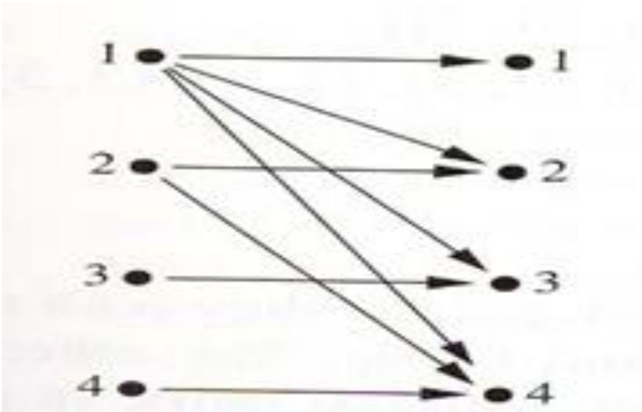
- Relations can be generalized to subsets of cartesian products of more than two sets

- Any subset of the Cartesian product of 3 sets is called a ternary relation
  - 'x and y are parents of z' $\subseteq$ People $\times$ People $\times$ People

- Any subset of the Cartesian product of k sets is called a k-ary relation

Functions as Relations

- A function f:A→B is a relation from A to B
- A relation from A to B is not always a function f:A→B
- Relations are generalizations of functions!



Functions as Relations

- Let A and B are two sets:
- R is relation between A and B.

- $R \subseteq A \times B$

- R will be function if
  - If **Every element of A** is related to some ***unique*** element of B

Relations on a Set

- A (binary) relation from a set *A* to itself is called a relation on the set *A*.
- A= {1,2,3,4}
- R={(a,b) | a divides b}

Describing Binary Relations

- A binary relation can be described using a list of pairs.

- Example:
- Among 6 people (Ali, Hamza, Umer, Asad, Haroon and Amir), Ali and Amir are brothers; and Umer, Asad and Haroon are brothers.
- $A \ = \{Ali, Hamza, Umer, Asad, Haroon, Amir\}$
- $Brotherhood \ = \{(x, y) |\ x \text{ is a brother of } y\}$
- $Brotherhood \ = \{(Ali, Amir), (Amir, Ali), (Umer, Asad), (Asad, Umer), (Umer, Haroon), (Haroon, Umer), (Asad, Haroon), (Haroon, Asad)\}$

Combining Relations

- Relations from A to B are subsets of $\mathrm{A} \times \mathrm{B}$, two relations from A to B can be combined in any way two sets can be combined.
- Let $A = \{ 1, 2, 3\}\ and\ B = \{1, 2, 3, 4\}$
- The relations $R_1 = \{(1,1), (2,2), (3,3)\}\ and$

  $$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$$
- $R_1 \cup R_2$
- $R_1 \cap R_2$
- $R_1 - R_2$
- $R_2 - R_1$

Combining Relations

- Let R be a relation from a set A to a set B and S be a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $\mathrm{a} \in A, c \in C$, and for which their exists an element $\mathrm{b} \in B$ such that $(\mathrm{a}, \mathrm{b}) \in R$ and $(\mathrm{b}, \mathrm{c}) \in S$. We denote the composite of R and S by $S{\circ}R$.
- What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Describing Binary Relations

- A binary relation can also be described using a matrix.
  - Rows are labeled with elements of A
  - Columns are labeled with elements of B.
- We write 1 in row a, column b if and only if $(a, b) \in R$; otherwise we write 0.

Describing Binary Relations

- $A = \{Ali, Hamza, Umer, Asad, Haroon, Amir\}$
- $Brotherhood = \{(x, y) \mid x \text{ is a brother of } y\}$
- $Brotherhood = \{(Ali, Amir), (Amir, Ali), (Umer, Asad), (Asad, Umer), (Umer, Haroon), (Haroon, Umer), (Asad, Haroon), (Haroon, Asad)\}$

	Ali	Hamza	Umer	Asad	Haroon	Amir
Ali	0	0	0	0	0	1
Hamza	0	0	0	0	0	0
Umer	0	0	0	1	1	0
Asad	0	0	1	0	1	0
Haroon	0	0	1	1	0	0
Amir	1	0	0	0	0	0

Describing Binary Relations

- A binary relation can also be described using a directed graph.
- A graph of a relation R ∈ A × B consists of two sets of vertices labeled by elements of A and B.
- A vertex **a** is connected to a vertex **b** with an edge (arc) if and only if (a, b) ∈ R.

Reflexivity

- Note: from now on we will consider only binary relations on A.
- That is such relations are subsets of $A \times A$.
- Reflexivity: A binary relation $\mathrm{R} \subseteq A \times A$ is said to be reflexive if $\forall a \in A, (a, a) \in R$
- $R = \{(a, b) | (a, b) \in Z \times Z, a \leq b\}$ is reflexive.

	a	b	c	d	e	...
a	1	.	.	.	.	.
b	.	1	.	.	.	.
c	.	.	1	.	.	.
d	.	.	.	1	.	.
e	.	.	.	.	1	.
...	.	.	.	.	.	1

Symmetricity

- Symmetricity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be symmetric if, $\forall a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$
- $R = \{(a, b) | (a, b) \in people \times people, a\ and\ b\ are\ siblings\}$ is symmetric, because if a is a sibling of b, then b is a sibling of a

	a	b	c	d	e	...
a	0	1	0	0	0	.
b	1	0	0	1	0	.
c	0	0	0	0	1	.
d	0	1	0	0	0	.
e	0	0	1	0	0	.
...	.	.	.	.	.	.

Transitivity:

- Transitivity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be transitive if, $\forall a, b, c \in A, (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R$
- $R = \{(a, b) | (a, b) \in Z \times Z, a \leq b\}$ is transitive, because if, $a \leq b$ and $b \leq c$ then $a \leq c$.

Anti-Symmetricity

- Anti-Symmetricity: A binary relation $\mathrm{R} \subseteq A \times A$ said to be anti-symmetric, if $\forall a, b \in A, (a, b) \in R \wedge (b, a) \in R \rightarrow a = b$
- The relation Motherhood ('x is the mother of y') is anti-symmetric, because if x is a mother of y, then y is not the mother of x.

	a	b	c	d	e	...
a	0	0	0	0	0	.
b	1	0	0	0	0	.
c	0	0	1	0	0	.
d	0	0	0	0	0	.
e	0	0	1	0	0	.
...	.	.	.	.	.	.

Equivalence Relations

- A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack
- R_1 = { (0,0), (1,1), (2,2), (3,3) }
  - R_1 has all the properties, thus, is an equivalence relation
- R_2 ={ (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
  - Not reflexive: (1,1) is missing
  - Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)
- R_3 = { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
  - Has all the properties, thus, is an equivalence relation
- R_4 ={ (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
  - Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)
- R_5 = { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
  - Not symmetric: (1,2) is present, but not (2,1)
  - Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

Examples

- Parenthood ('x is the parent of y')
- Brotherhood ('x is the brother of y')
- Neighborhood ('x is the neighbor of y')
- Ownership ('x is the owner of y‘)
- 'x divides y'
- $x \leq y$
- $x = y$
- $x < y$

Chapter Reading and Exercise

Chapter # 9

Topic # 9.1

Q-1,2,3,10,11,18,30,32

Topic # 9.3

Q-1,2,3,4,5,6,7,8,13-c,14,23-28