CSD101 - Discrete Structures
(Discrete Mathematics)
Fall 2016

Lecture - 11

Sequences
and
Summations

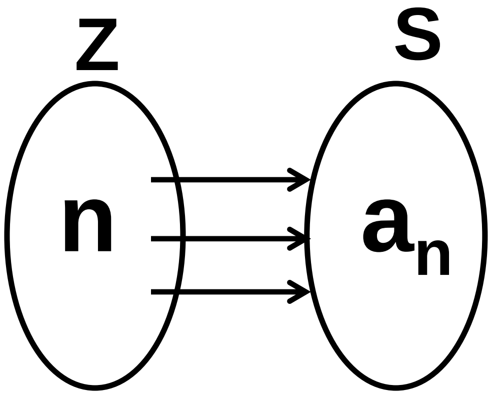
Sequences

- A sequence is a discrete structure used to represent an ordered list of elements e.g. 1, 2, 3, 4, 5 and 1, 3, 9, 27, 81, ….

Sequences

- A **sequence** is a function from a subset of the set integers **Z** (usually the set {0,1,2,…} or the set {1,2,3,…}) to a set **S**.

- The notation a_n denotes the image of the integer n.
- a_n : a *term* of the sequence
- $\{a_n\}$: entire sequence
  - Same notation as sets!



Sequences

- Consider the sequence $\{a_n\}$, where $a_n = 1/n$.
  - The list of the terms of this sequence beginning with a_1:

    $a_1, a_2, a_3, a_4, \ldots$

    $\{1, 1/2, 1/3, 1/4, \ldots\}$

- Consider the sequence $\{a_n\}$, where $a_n = 3n$.
  - The list of the terms of this sequence beginning with a_1:

    $\{3, 6, 9, 12, \ldots\}$

Geometric Progression

A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \dots., ar^n, \dots$$

Where the **initial term *a*** and the **common ratio *r*** are real numbers.

General Term of Geometric Progression

- Let **a** be the first term and **r** be the common ratio of a geometric sequence. Then the sequence is

$$a, ar, ar^2, ar^3, \ldots$$

- If a_n, for $\mathrm{n} \geq 1$, represents the terms of the sequence then

$a_1 =$ first term $= a = ar^{1-1}$

$a_2 =$ second term $= ar = ar^{2-1}$

$a_3 =$ third term $= ar^2 = ar^{3-1}$

By symmetry

$a_n =$ nth term $= ar^{n-1}$ for all integers $\mathrm{n} \geq 1$.

Geometric Progression (Example)

- Is $\{2.(5)^{n-1}\}$ geometric progression? If yes then what will be sequence, initial term and common ratio?

  2,10,50,250,…

  Yes, a=2 and r=5

- Is $\{6.(^1/_3)^{n-1}\}$ geometric progression?

  6,2,2/3,2/9,…

  Yes, a=6 and r=1/3

Geometric Progression (Example

- Find the 8th term of the following geometric sequence

$$4, 12, 36, 108, \ldots$$

Arithmetic Progression

- An **arithmetic progression** is a sequence of the form

$$a, a + d, a + 2d, \ldots . ., a + nd, \ldots$$

- Where the initial term ***a*** and the **common difference *d*** are real numbers.

General Term of Arithmetic Progression

- Let **a** be the first term and **d** be the common difference of an arithmetic sequence. Then the sequence is

$$a, a + d, a + 2d, a + 3d, \ldots$$

- If a_n, for $\mathrm{n} \geq 1$, represents the terms of the sequence then

  $a_1 =$ first term $= a = a + (1 - 1)d$

  $a_2 =$ second term $= a + d \; = a + (2 - 1)d$

  $a_3 =$ third term $= \; a + 2d = a + (3 - 1)d$

  By symmetry

  $a_n =$ nth term $= a + (n - 1)d$ for all integers $\mathrm{n} \geq 1$.

Arithmetic Progression (Example)

- Is $\{4n - 5\}$ Arithmetic progression?

  -1,3,7,11,…

  Yes, a=-1 and d=4

- Is $\{10 - 3n\}$ Arithmetic progression?

  7,4,1,-2,…

  Yes, a=7 and d=-3

Arithmetic Progression (Example)

- Find the 20th term of the arithmetic sequence

$$3, 9, 15, 21, \dots$$

- Which term of the arithmetic sequence

$$4, 1, -2, \dots, is\ -77$$

Determining the Sequence Formula

- Given values in a sequence, how do you determine the formula?
- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence repeat itself (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

Sequences (Example)

- Find a formula for the following sequence.

$$1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \ldots$$

Solution:

The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time.

Sequences (Example)

- Find a formula for the following sequence.

$$1, 1/2, 1/4, 1/8, 1/16, \ldots$$

Solution:

$\{1/2^{n-1}\}$

It is a geometric progression.

a=1 and r=1/2

Sequences (Example)

- Find formula for the following sequence.

$$1, 3, 5, 7, 9, \ldots$$

Solution:

$\{2n - 1\}$

It is a arithmetic progression.

a=1 and d=2

Sequences (Example)

- Find formula for the following sequence.

$$1, -1, 1, -1, 1, \ldots$$

Solution:

$\{(-1)^{n-1}\}$

It is a geometric progression.

a=1 and r=-1

Sequences (Example)

- How can you produce the terms of the following sequence?

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$$

Solution:

A rule for generating this sequence is that integer n appears exactly n times.

Sequences (Example)

- How can you produce the terms of the following sequence?

$$5, 11, 17, 23, 29, 35, 41, \ldots$$

Solution:

A rule for generating this sequence is $6n - 1$.

It is an arithmetic progression.

a=5 and d=6

Sequences (Example)

- Find a formula for the following sequence.

$$15, 8, 1, -6, -13, -20, -27, \ldots$$

Solution:

Each term is 7 less than the previous term.

$a_n = 22 - 7n$

Useful Sequences

*n*th Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Sequences (Example)

- Find a formula for the following sequence?

$$2, 16, 54, 128, 250, 432, 686, \ldots$$

Solution:

Each term is twice the cube of n.

$a_n = 2 * n^3$

Sequences (Example)

- Find formula for the following sequence.

$$1, 7, 25, 79, 241, 727, 2185, \ldots$$

Solution:

Compare it to $\{3^n\}$.

$\{3^n - 2\}$

Summations

- The sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$ is:

- $a_m, a_{m+1}, \ldots, a_n$
- $\sum_{j=m}^{n} a_j$
- $\sum_{m \leq j \leq n} a_j$, where $\sum$ donates **summation** and j is the **index of summation**.

- m is **lower limit** and n is **upper limit.**

Summations

- A summation:

$$\sum_{j=m}^{n} a_j$$

is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
     sum += a(j);
```

Summations (Example)

Express the sum of the first 100 terms of the sequence {1/n} for n=1,2,3,… .

Solution:

$$\sum_{n=1}^{100} 1/n$$

Summations (Example)

What is the value of $\sum_{i=1}^{3} i^2$?

Solution:

$$\sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14$$

More Summations (Example)

- $\sum_{k=1}^{5}(k+1) = 2 + 3 + 4 + 5 + 6 = 20$

- $\sum_{k=0}^{4}(-2)^k = (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

- $\sum_{k=1}^{10} 3 = 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30$

More Summations (Example)

Evaluate $\sum_{k=1}^{10}(2^k - 2^{k-1}) = ?$

Solution:

$$\sum_{k=1}^{10}(2^k - 2^{k-1}) = (2^1 - 2^0) + (2^2 - 2^1) + (2^3 - 2^2) + (2^4 - 2^3) + (2^5 - 2^4) + (2^6 - 2^5) + (2^7 - 2^6) + (2^8 - 2^7) + (2^9 - 2^8) + (2^{10} - 2^9)$$

$$= -1 + 2^{10} = -1 + 1024 = 1023$$

Shifting the Index of Summation

- Useful in case of sum.

- $\sum_{j=1}^{5} j^2$ shift the index of summation from 0 to 4 rather than from 1 to 5.

Properties of Summations

$$\sum_{k=m}^{n}(a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k; \quad a_k, b_k \in R$$

$$\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k \quad c \in \mathrm{R}$$

$$\sum_{k=1}^{n} c = c + c + \cdots + c = nc$$

Example

$$Solve\ \ 3\sum_{k=1}^{n}(2k-3)+\sum_{k=1}^{n}(4-5k)$$

Double Summations

- Like a nested for loop
- $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$

Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
     for ( int j = 1; j <= 3; j++ )
          sum += i*j;
```

Double Summations

- $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$

Example

$Solve \sum_{i=1}^{3} \sum_{j=1}^{2} (i-j).$

Some Useful Summations

TABLE for Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Example

Find $\sum_{k=50}^{100} k^2$.

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2,$

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

because $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$

Example

Find $\sum_{k=100}^{200} k$.

Find $\sum_{k=99}^{200} k^3$.

Exercise Questions

Chapter # 2

Topic # 2.4

Questions 1, 2, 4, 25, 26, 29,30,31, 32, 33, 34, 39, 40