Matrices, types, determinants and inverses using adjoints

Linear Algebra - 24 DS

AH Sheikh

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Introduction to Matrices

- The word "matrix" was first used by James Sylvester (1814-1897)
- Arthur Cayley (1821-1895) developed matrix theory
- Used in linear transformations, high-speed computers, and various disciplines
- Determinants: Seki Kowa (1642-1708) and Leibniz (1646-1716), applied by Cramer (1704-1752)

Definition: A rectangular array of numbers:

$$\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 4 \\ 3 & 2 & 6 \\ 4 & 1 & -1 \end{bmatrix}$$

Matrix Notation

General $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- $m \times n$ is the **order** of the matrix
- Horizontal lines: **rows**, Vertical lines: **columns**
- Entries: a_{ij} (element in ith row, jth column)
- Notation: $A = [a_{ij}]_{m\times n}$ or $A = [a_{ij}]$

Example: In $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 4 & 7 \end{bmatrix}$, element 7 is at a_{23}

Special Types of Matrices

1. **Row Matrix/Vector:** $1 \times n$ matrix
   Example: $[1 \ \ -1 \ 3 \ 4]$
2. **Column Matrix/Vector:** $m \times 1$ matrix
   Example: $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$
3. **Rectangular Matrix:** $m \neq n$
   Example: $\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$
4. **Square Matrix:** $m = n$
   Example: $\begin{bmatrix} 2 & 5 \\ -1 & 6 \end{bmatrix}$

Diagonal and Scalar Matrices

Diagonal Matrix: Square matrix with $a_{ij} = 0$ for $i \neq j$

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Scalar Matrix: Diagonal matrix with all diagonal elements equal

Example: $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Identity Matrix: Scalar matrix with diagonal elements $= 1$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix and Equal Matrices

Null/Zero Matrix: All elements are zero

Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, denoted by O

Equal Matrices: $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal if:

$$a_{ij} = b_{ij} \quad \text{for all } i = 1, \ldots, m; \; j = 1, \ldots, n$$

Example:

$$\begin{bmatrix} x+3 & 1 \\ 3 & y+4 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}$$

Then $x + 3 = 5 \Rightarrow x = 2$ and $y + 4 = 2 \Rightarrow y = -2$

Matrix Addition

Two matrices are **conformable for addition** if they have the same order. If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then:

$$A + B = [a_{ij} + b_{ij}]_{m\times n}$$

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 4 & -2 \\ 3 & 1 & 1 \end{bmatrix}$$

Matrix Subtraction

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then:

$$A - B = A + (-B) = [a_{ij} - b_{ij}]_{m\times n}$$

Example:

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ -2 & 4 & -3 \end{bmatrix}$$

Scalar Multiplication

If $A = [a_{ij}]_{m\times n}$ and k is a scalar, then:

$$kA = [ka_{ij}]_{m\times n}$$

Properties:

- $r(sA) = (rs)A$
- $(r+s)A = rA + sA$
- $r(A+B) = rA + rB$
- $A + A + \cdots + A$ (n times) $= nA$

Example: If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & -6 \\ 9 & 12 \end{bmatrix}$

Matrix Multiplication

Two matrices A and B are **conformable for multiplication** if:

$$\text{Number of columns of } A = \text{Number of rows of } B$$

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $C = AB = [c_{ik}]_{m\times p}$ where:

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}$$

Note: In general, $AB \neq BA$

Matrix Multiplication Example

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}_{3\times 2}$

Then

$$AB = \begin{bmatrix} 2\times 1 + 0\times 2 + 1\times 1 & 2\times 1 + 0\times 3 + 1\times 0 \\ 1\times 1 + 4\times 2 + 2\times 1 & 1\times 1 + 4\times 3 + 2\times 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 11 & 13 \end{bmatrix}_{2\times 2}$$

And $BA = \begin{bmatrix} 1\times 2 + 1\times 1 & 1\times 0 + 1\times 4 & 1\times 1 + 1\times 2 \\ 2\times 2 + 3\times 1 & 2\times 0 + 3\times 4 & 2\times 1 + 3\times 2 \\ 1\times 2 + 0\times 1 & 1\times 0 + 0\times 4 & 1\times 1 + 0\times 2 \end{bmatrix} =$

$$\begin{bmatrix} 3 & 4 & 3 \\ 7 & 12 & 8 \\ 2 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Clearly $AB \neq BA$

Transpose of a Matrix

If $A = [a_{ij}]_{m\times n}$, then the **transpose** $A^T = [a_{ji}]_{n\times m}$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

Properties:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(kA)^T = kA^T$

Note: If $(A^T) = A$, then matrix is called **Symmetric.**

Determinant of a 2×2 Matrix

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$|A| = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Examples:

$$\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)(3) - (-1)(4) = 6 + 4 = 10$$

$$\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = (1)(8) - (4)(2) = 8 - 8 = 0$$

For 1×1 matrix: $|[a_{11}]| = a_{11}$

Singular and Non-Singular Matrices

A square matrix A is:

- **Singular** if $|A| = 0$
- **Non-singular** if $|A| \neq 0$

Examples:

- $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, $|A| = 10 \neq 0$ (non-singular)
- $B = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$, $|B| = 0$ (singular)

Adjoint and Inverse of 2×2 Matrix

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

Adjoint:

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse: If A is non-singular ($|A| \neq 0$), then:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: $A = \begin{bmatrix} 5 & 3 \\ 1 & 1 \end{bmatrix}$, $|A| = 5 - 3 = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$$

Verify: $AA^{-1} = A^{-1}A = I_2$

Properties of Matrix Operations

For matrices A, B, C and scalars c, d:

1. Commutative for addition: $A + B = B + A$
2. Associative for addition: $(A + B) + C = A + (B + C)$
3. Associative for scalar multiplication: $(cd)A = c(dA)$
4. Additive identity: $A + O = O + A = A$
5. Multiplicative identity: $IA = AI = A$
6. Distributive: $c(A + B) = cA + cB$, $(c + d)A = cA + dA$
7. Associative for multiplication: $A(BC) = (AB)C$
8. Left distributive: $A(B + C) = AB + AC$
9. Right distributive: $(A + B)C = AC + BC$
10. $c(AB) = (cA)B = A(cB)$

Matrix Multiplication Properties - Example

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 \times 1 + 0 \times 2 + 1 \times 1 & 2 \times 1 + 0 \times 3 + 1 \times 2 & 2 \times 0 + 0 \times 1 + 1 \times 3 \\ -1 \times 1 + 4 \times 2 + 2 \times 1 & -1 \times 1 + 4 \times 3 + 2 \times 2 & -1 \times 0 + 4 \times 1 + 2 \times 3 \\ 3 \times 1 + 1 \times 2 + 3 \times 1 & 3 \times 1 + 1 \times 3 + 3 \times 2 & 3 \times 0 + 1 \times 1 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 3 \\ 9 & 15 & 10 \\ 8 & 12 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 \times 2 + 1 \times (-1) + 0 \times 3 & 1 \times 0 + 1 \times 4 + 0 \times 1 & 1 \times 1 + 1 \times 2 + 0 \times 3 \\ 2 \times 2 + 3 \times (-1) + 1 \times 3 & 2 \times 0 + 3 \times 4 + 1 \times 1 & 2 \times 1 + 3 \times 2 + 1 \times 3 \\ 1 \times 2 + 2 \times (-1) + 3 \times 3 & 1 \times 0 + 2 \times 4 + 3 \times 1 & 1 \times 1 + 2 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 13 & 11 \\ 9 & 11 & 14 \end{bmatrix}$$

Clearly $AB \neq BA$

Determinant of 3×3 Matrix

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$:

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Or:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example:

$$\begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$$= 1(6+3) + 2(-4-4) + 3(6-12) = 9 - 16 - 18 = -25$$

Minor and Cofactor

For a square matrix $A = [a_{ij}]_{n\times n}$:
Minor M_{ij}: Determinant of $(n-1)\times(n-1)$ matrix obtained by deleting ith row and jth column of A.
Cofactor A_{ij}:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Example: $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

$M_{12} = \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} = (-2)(2) - (1)(4) = -4 - 4 = -8$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3(-8) = 8$

General Definition of Determinant

For $A = [a_{ij}]_{n\times n}$:

Row expansion:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad \text{for any fixed } i$$

Column expansion:

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad \text{for any fixed } j$$

Example: For $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & 1 \\ 4 & -3 & 2 \end{bmatrix}$, expanding by first row:

$$|A| = 1 \cdot A_{11} + (-2) \cdot A_{12} + 3 \cdot A_{13}$$

$$= 1 \cdot (-1)^{1+1} \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix} + (-2) \cdot (-1)^{1+2} \begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + 3 \cdot (-1)^{1+3} \begin{vmatrix} -2 & 3 \\ 4 & -3 \end{vmatrix}$$

$$= 1(1)(6+3) + (-2)(-1)(-4-4) + 3(1)(6-12) = 9 + (-16) - 18 = -25$$

Properties of Determinants

1. $|A| = |A^T|$
2. Interchanging two rows/columns changes sign: $|A'| = -|A|$
3. Two identical rows/columns $\Rightarrow |A| = 0$
4. Row/column of zeros $\Rightarrow |A| = 0$
5. Multiplying a row/column by k: $|A'| = k|A|$
6. If a row/column has two terms: $|A| = |B| + |C|$
7. Adding k times a row/column to another doesn't change $|A|$
8. For triangular matrix: $|A| =$ product of diagonal entries

Example: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \times 4 \times 6 = 24$

Properties Examples

Property 1: $|A| = |A^T|$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Property 5: Multiplying row by k:

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k(ad - bc) = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Property 6: Row with two terms:

$$\begin{vmatrix} a+b & c \\ d & e \end{vmatrix} = \begin{vmatrix} a & c \\ d & e \end{vmatrix} + \begin{vmatrix} b & c \\ d & e \end{vmatrix}$$

Example Without Expansion

Show $\begin{vmatrix} x+a & x+b & c \\ x+b & x+c & a \\ x+c & x+a & b \end{vmatrix} = 0$

Solution: $C_2 \leftarrow C_2 - C_1$:

$$\begin{vmatrix} x+a & b-a & c \\ x+b & c-b & a \\ x+c & a-c & b \end{vmatrix} = x\begin{vmatrix} 1 & b-a & c \\ 1 & c-b & a \\ 1 & a-c & b \end{vmatrix} + \begin{vmatrix} a & b-a & c \\ b & c-b & a \\ c & a-c & b \end{vmatrix}$$

First determinant has identical columns (1st and 2nd after adding), so $= 0$

Second: $C_1 \leftarrow C_1 + C_2$:

$$\begin{vmatrix} a+b-a & b-a & c \\ b+c-b & c-b & a \\ c+a-c & a-c & b \end{vmatrix} = \begin{vmatrix} b & b-a & c \\ c & c-b & a \\ a & a-c & b \end{vmatrix}$$

C_1 and C_2 are identical, so $= 0$. Thus original determinant $= 0$.

Adjoint of $n \times n$ Matrix

For $A = [a_{ij}]_{n\times n}$, the **cofactor matrix** is $[A_{ij}]$ where A_{ij} is the cofactor of a_{ij}.
Adjoint: Transpose of cofactor matrix

$$\text{adj}(A) = [A_{ij}]^T = [A_{ji}]$$

Example: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$A_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$, $A_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(0-1) = 1$,

$A_{13} = \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 0 - 2 = -2$

$A_{21} = -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -(0-2) = 2$, $A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1$,

$A_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -(1-0) = -1$

Inverse of $n \times n$ Matrix

For non-singular A ($|A| \neq 0$):

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

Example continued: $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$|A| = 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = 1(2-1)+0+2(0-2) = 1-4 = -3$

$A^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & 2 & -4 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

Verify: $AA^{-1} = I_3$

Properties of Inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^{-1})^{-1} = A$
- $|A^{-1}| = \frac{1}{|A|}$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$ for $k \neq 0$

Example: Verify $(AB)^{-1} = B^{-1}A^{-1}$ for $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, $B = \begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$

$$AB = \begin{bmatrix}4 & 7\\10 & 15\end{bmatrix},\ |AB| = 60 - 70 = -10,$$

$$(AB)^{-1} = -\frac{1}{10}\begin{bmatrix}15 & -7\\-10 & 4\end{bmatrix} = \begin{bmatrix}-1.5 & 0.7\\1 & -0.4\end{bmatrix}$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\1.5 & -0.5\end{bmatrix},$$

$$B^{-1} = \frac{1}{5}\begin{bmatrix}3 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}0.6 & -0.2\\-0.2 & 0.4\end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix}0.6 & -0.2\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix} = \begin{bmatrix}-1.5 & 0.7\end{bmatrix} = (AB)^{-1}$$