Row Operations, Echelon form and Rank

Linear Algebra - 24 DS

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Elementary Row Operations

Three types:

1. $R_i \leftrightarrow R_j$: Interchange rows
2. $R_i \to kR_i$: Multiply row by non-zero k
3. $R_i \to R_i + kR_j$: Add k times row j to row i

Elementary Column Operations (similar notations with C)
Two matrices are **equivalent** ($A \sim B$) if B can be obtained from A by elementary operations.
Augmented matrix: For system $AX = B$:

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

Echelon Form

Matrix is in **(row) echelon form** if:

1. In each non-zero row, the number of leading zeros is greater than in preceding row
2. All non-zero rows precede zero rows (if any)
3. First non-zero entry (leading entry) in each row is 1

Examples:

$$\begin{bmatrix} 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are in echelon form.

Reduced Echelon Form: Additional condition: Leading 1 is only non-zero in its column.

Reducing to Echelon Form - Example

Reduce $A = \begin{bmatrix} 2 & 3 & 1 & 9 \\ 1 & -1 & 2 & -3 \\ 3 & 1 & 3 & 2 \end{bmatrix}$ to echelon form.

$R_1 \leftrightarrow R_2$: $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 3 & 1 & 9 \\ 3 & 1 & 3 & 2 \end{bmatrix}$, $\quad R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - 3R_1 :$

$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 5 & -3 & 15 \\ 0 & 4 & -3 & 11 \end{bmatrix}$ $R_2 \leftarrow \frac{1}{5}R_2 :$ $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -\frac{3}{5} & 3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$,

$\mathrm{R}_3 \leftarrow R_3 - 4R_2 :$ $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -\frac{3}{5} & 3 \\ 0 & 0 & -\frac{3}{5} & -1 \end{bmatrix}$ $\quad R_3 \leftarrow -\frac{5}{3}R_3 :$

$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -\frac{3}{5} & 3 \\ 0 & 0 & 1 & \frac{5}{3} \end{bmatrix}$ (Echelon form)

Further reduction gives reduced echelon form.

Reduced Row-Echelon Form Example 2

Problem: Reduce the following 3×3 matrix to RREF:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

Step 1: Start with A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$$

Step 2: $R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Reduced Row Echelon Form - Example 2 Continued

Step 3: $R_3 \leftarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 4: $R_1 \leftarrow R_1 - 2R_2$ (Back substitution)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Final RREF:

$$\boxed{\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Reduced Row Echelon Form - Example 2 Continued

Properties:

- Leading 1's in first two columns
- Rows of zeros at the bottom $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- Matrix is singular ($rank = 2$, not 3)

Example 1: Finding Inverse using Row Operations

For non-singular A, to find A^{-1}:

$$[A|I] \sim [I|A^{-1}]$$

by elementary row operations. **Example:** $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 4 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \; R_1 \leftrightarrow R_3:$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & 1 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 2 & 5 & -1 & 1 & 0 & 0 \end{array}\right] \; R_2 \leftarrow R_2 - 3R_1,\; R_3 \leftarrow R_3 - 2R_1:$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 0 & 0 & 1 \\ 0 & -2 & 8 & 0 & 1 & -3 \\ 0 & 1 & 3 & 1 & 0 & -2 \end{array}\right]$$

Continue until left side becomes I, right side becomes A^{-1}.

$$\begin{bmatrix} -\frac{6}{7} & \frac{8}{7} & -\frac{2}{7} \end{bmatrix}$$

Example 2: Finding Inverse Using Row Operations

Problem: Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

using row operations.

Solution to Example 2

Step 1: Augment with Identity Matrix

Form the augmented matrix $[A|I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Goal: Transform left side to I_3; right side becomes A^{-1}.

Solution to Example 2

Step 2: Make $a_{31} = 0$

$R_3 \leftarrow R_3 - 5R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

Solution to Example 2

Step 3: Make $a_{32} = 0$

$R_3 \leftarrow R_3 + 4R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Now the left side is upper triangular.

Step 4: Make $a_{23} = 0$ and $a_{13} = 0$

1. $R_2 \leftarrow R_2 - 4R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

2. $R_1 \leftarrow R_1 - 3R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Solution to Example 2

Step 5: Make $a_{12} = 0$

$R_1 \leftarrow R_1 - 2R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Left side is now I_3.

Solution to Example 2

Step 6: Extract Inverse

The inverse matrix is:

$$A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

Verification

Check $AA^{-1} = I$:

$$AA^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3 \quad \checkmark$$

Similarly, $A^{-1}A = I_3$ holds.

Rank of a Matrix

Definition

The **rank** of a matrix is the number of non-zero rows in its reduced echelon form.

Rank of a Matrix

Example: How to find rank

Example: Find rank of $A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 7 & 7 \\ 3 & 1 & 12 & 11 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - 3R_1$: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{bmatrix}$

$R_3 \leftarrow R_3 - 2R_2$: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftarrow \frac{1}{2}R_2$: $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (Echelon form)

Number of non-zero rows $= 2$, so $\text{rank}(A) = 2$ When rank of matrix is number of rows in matrix, then matrix is called **Full Rank Matrix**.