System Solving using Different Methods

Linear Algebra - 24 DS

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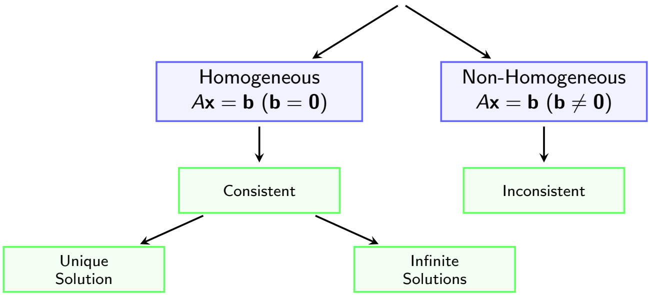
Linear Systems - Types

- **Homogeneous:** $AX = 0$ (always consistent, has trivial solution $X = 0$)
- **Non-homogeneous:** $AX = B$, $B \neq 0$
- **Consistent:** Has solution(s)
- **Inconsistent:** No solution

Example systems:

1. $\begin{cases} 2x_1 + 5x_2 - x_3 = 5 \\ 3x_1 + 4x_2 + 2x_3 = 11 \\ x_1 + 2x_2 - 2x_3 = -3 \end{cases}$ (Consistent, unique solution)
2. $\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 - x_2 + 7x_3 = 11 \\ 3x_1 + 5x_2 + 4x_3 = -3 \end{cases}$ (Consistent, infinite solutions)
3. $\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 - 6x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 4x_3 = -3 \end{cases}$ (Inconsistent)

Types of Linear Systems - Flow Chart



- **Homogeneous**: Always has at least trivial solution $\mathbf{x} = \mathbf{0}$
- **Non-homogeneous**: May be consistent or

Consistency Conditions

For $AX = B$:
Let $r = \text{rank}(A)$, $r' = \text{rank}([A|B])$, $n =$ number of variables.

1. **Consistent with unique solution:** $r = r' = n$
2. **Consistent with infinite solutions:** $r = r' < n$
3. **Inconsistent:** $r \neq r'$

Homogeneous system $AX = 0$:

- Always consistent (trivial solution $X = 0$)
- Non-trivial solution exists iff $|A| = 0$
- If $|A| \neq 0$, only trivial solution

Solving Linear Systems - **Matrix Method**

For $AX = B$, if $|A| \neq 0$:

$$X = A^{-1}B$$

Example: Solve $\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ 2x_1 - 3x_2 + 2x_3 = 6 \\ 2x_1 + 2x_2 + x_3 = 5 \end{cases}$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$|A| = 1(-3-4) - (-2)(2-4) + 1(4+6) = -7 - 4 + 10 = -1$$

$$A^{-1} = \frac{1}{-1}\text{adj}(A) = -\begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution: $x_1 = 1$, $x_2 = 2$, $x_3 = -1$

Solving Linear System : **Cramer's Rule**

For $AX = B$ with $|A| \neq 0$: Let $A_i =$ matrix A with column i replaced by B. Then:

$$x_i = \frac{|A_i|}{|A|} \quad \text{for } i = 1, 2, \ldots, n$$

Example: Solve $\begin{cases} 3x_1 + x_2 - x_3 = -4 \\ x_1 + x_2 - 3x_3 = 4 \\ -x_1 + 2x_2 - x_3 = -1 \end{cases}$

$$|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -3 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+6) - 1(-1-3) - 1(2+1) = 15 + 4 - 3 = 16$$

$$x_1 = \frac{\begin{vmatrix} -4 & 1 & -1 \\ 4 & 1 & -3 \\ -1 & 2 & -1 \end{vmatrix}}{16} = \frac{-4(-1+6) - 1(-4-3) - 1(8+1)}{16} = \frac{-20+7-9}{16} = \frac{-22}{16} = -\frac{11}{8}$$

Similarly: $x_2 = \frac{8}{16} = \frac{1}{2}$, $x_3 = \frac{18}{16} = \frac{9}{8}$

Solving Linear System : **Substitution Method**

Substitution Method:

1. Solve one equation for one variable in terms of the others.
2. Substitute this expression into the remaining equations.
3. Repeat until a single equation in one variable remains.
4. Back-substitute to find all variables.

Idealy, it is used : when one variable has coefficient 1 or -1.

Substitution Method - 2×2 System

Example: Solve

$$\begin{cases} x + 2y = 5 \\ 3x - y = 1 \end{cases}$$

1. Solve one eq. for one var. in terms of of ther other; So from first equation, we get $x = 5 - 2y$.
2. Substitute into second equation, we get

$$3(5 - 2y) - y = 1 \text{ OR } 15 - 6y - y = 1 \quad \Rightarrow \quad 15 - 7y = 1$$

$$-7y = -14 \quad \Rightarrow \quad y = 2$$

3. Repeat (Only two equations, no need to repeat)
4. Back-substitute: $x = 5 - 2(2) = 1$

$$\boxed{x = 1, \; y = 2}$$

Substitution Method - 3×3 System

Example: Solve

$$\begin{cases} x + y + z = 6 \\ y - z = 1 \\ 2x + y = 7 \end{cases}$$

1. From second: $y = 1 + z$
   Step 2: From third: $2x + (1 + z) = 7 \Rightarrow 2x + z = 6 \Rightarrow x = 3 - \frac{z}{2}$
2. Substitute first:
   $$\left(3 - \frac{z}{2}\right) + (1 + z) + z = 6$$
   $$4 + \frac{3z}{2} = 6 \quad \Rightarrow \quad \frac{3z}{2} = 2 \quad \Rightarrow \quad z = \frac{4}{3}$$
3. Back-substitution, we will get: $y = 1 + \frac{4}{3} = \frac{7}{3}$, $x = 3 - \frac{2}{3} = \frac{7}{3}$

$$\boxed{x = \frac{7}{3},\ y = \frac{7}{3},\ z = \frac{4}{3}}$$

Solving Linear System : **Elimination Method**

Elimination Method (Also called Gaussian Elimination)

Elimination Method:

1. Multiply equations by constants to align coefficients.
2. Add/subtract equations to eliminate one variable.
3. Repeat until one equation in one variable remains.
4. Back-substitute to find all variables.

Goal: Transform system into upper triangular form.

Elimination Method - 2×2 System

Example: Solve

$$\begin{cases} 2x + 3y = 8 \\ 3x - y = 1 \end{cases}$$

1. Multiply second eqn by 3: $9x - 3y = 3$
2. Add to first eqn:

$$(2x + 3y) + (9x - 3y) = 8 + 3$$

$$11x = 11 \quad \Rightarrow \quad x = 1$$

3. Repeat (no need to repeat, as there are only two equations)
4. Back-substitute: $3(1) - y = 1 \Rightarrow y = 2$

$$\boxed{x = 1,\ y = 2}$$

Elimination Method - 3×3 System

Example: Solve

$$\begin{cases} x + y - z = 2 \\ 2x - y + z = 3 \\ x + 2y + z = 7 \end{cases}$$

1. Add eqn1 and eqn2:
   $(x + y - z) + (2x - y + z) = 2 + 3 \Rightarrow 3x = 5 \Rightarrow x = \frac{5}{3}$
2. Add eqn1 and eqn3:
   $(x + y - z) + (x + 2y + z) = 2 + 7 \Rightarrow 2x + 3y = 9$
   Substitute $x = \frac{5}{3}$:
   $2(\frac{5}{3}) + 3y = 9 \Rightarrow \frac{10}{3} + 3y = 9 \Rightarrow 3y = \frac{17}{3} \Rightarrow y = \frac{17}{9}$
3. From eqn1:
   $\frac{5}{3} + \frac{17}{9} - z = 2 \Rightarrow \frac{15}{9} + \frac{17}{9} - z = 2 \Rightarrow \frac{32}{9} - z = 2 \Rightarrow z = \frac{14}{9}$

$$\boxed{x = \frac{5}{3},\ y = \frac{17}{9},\ z = \frac{14}{9}}$$

Homogeneous Systems - Non-trivial Solutions

For $AX = 0$: Non-trivial solution exists iff $|A| = 0$ **Example:** Find λ for non-trivial solution: $\begin{cases} x+y+z=0 \\ 2x+y-\lambda z=0 \\ x+2y-2z=0 \end{cases}$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 1(-2+2\lambda) - 1(-4+\lambda) + 1(4-1) =$
$-2+2\lambda+4-\lambda+3 = \lambda+5$

For non-trivial solution: $|A| = 0 \Rightarrow \lambda+5=0 \Rightarrow \lambda=-5$

When $\lambda=-5$, solve: $\begin{cases} x+y+z=0 \\ 2x+y+5z=0 \\ x+2y-2z=0 \end{cases}$

Solution: $x=-3t$, $y=t$, $z=t$ for any $t \in \mathbb{R}$