

# Determinant and System Solving using elementary row operations

Linear Algebra - 24 DS

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# Basic Theorems

## Theorem (Zero Row/Column)

*If  $A$  has a row of zeros or a column of zeros, then  $\det(A) = 0$ .*

## Proof.

Expand along the zero row/column:

$$\det(A) = 0 \cdot C_1 + 0 \cdot C_2 + \cdots + 0 \cdot C_n = 0.$$



## Theorem (Determinant of Transpose)

$$\det(A) = \det(A^T).$$

# Effect of Elementary Row Operations

## Theorem (How row operations changes determinant)

Let  $A$  be  $n \times n$ .

- (a) If  $B$  results from multiplying a row/column of  $A$  by  $k$ , then  $\det(B) = k \det(A)$ .
- (b) If  $B$  results from interchanging two rows/columns, then  $\det(B) = -\det(A)$ .
- (c) If  $B$  results from adding a multiple of one row/column to another, then  $\det(B) = \det(A)$ .

# How row operations affect determinant

## Note

*det(I<sub>3</sub>) = 1, then what will be det(E) elementary matrices obtained from identity*

Operation	Matrix $B$	Determinant
Multiply row by $k$	$\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\det(B) = k \det(A)$
Interchange any two rows	$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\det(B) = -\det(A)$
Add multiple of row to another	$\begin{bmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$	$\det(B) = \det(A)$

# Elementary Matrices

## Theorem (Theorem 2.2.4)

Let  $E$  be an  $n \times n$  elementary matrix.

- (a) If  $E$  from multiplying a row of  $I_n$  by  $k$ , then  $\det(E) = k$ .
- (b) If  $E$  from interchanging two rows of  $I_n$ , then  $\det(E) = -1$ .
- (c) If  $E$  from adding a multiple of one row to another, then  $\det(E) = 1$ .

## Example

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 3, \quad \det \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -1, \quad \det \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

# Proportional Rows or Columns

## Theorem ((Same Rows?))

*If  $A$  has two proportional rows or two proportional columns, then  $\det(A) = 0$ .*

## Example

Each has determinant zero:

$$\begin{pmatrix} -1 & 4 \\ -2 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -4 \\ 7 & 5 & 3 \end{pmatrix}$$

# Method: Evaluation by Row Reduction

- 1 Reduce matrix to upper triangular form using elementary row operations.
- 2 Keep track of how operations affect the determinant.
- 3 Determinant of triangular matrix = product of diagonal entries.
- 4 Adjust determinant based on operations performed.

# Rows versus Columns

Determinant can be computed using operations on columns too.

## Example 1: Using row operations

Evaluate  $\det(A)$  where  $A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{Row interchange}) \quad (1)$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad (\text{Factor 3 from row 1}) = -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad (\text{Add } -2 \times \text{row 1 to row 3})$$

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \quad (\text{Add } -10 \times \text{row 2 to row 3}) =$$

$$(-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = (-3)(-55)(1) = 165. \quad (2)$$

## Example 2: Using Column Operations

Compute  $\det(A)$  where

$$A = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix}.$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 3 & -4 \end{vmatrix} \quad (\text{Add } -3 \times \text{col 1 to col 4}) \\ &= (1)(1) \begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix} = 2(-4) - (-5)(3) = 7. \end{aligned}$$

## Example 3: Combination with Cofactor Expansion

Evaluate  $\det(A)$  where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

**Warning!!!**

Not feasible for  
machine/computer  
implementation.

- Strategy: Use row operations to introduce zeros, then cofactor expansion.
- Add suitable multiples of row 2 to other rows to simplify.
- Result: Reduced matrix allows easier determinant calculation.

# Linear Systems - Types

- **Homogeneous:**  $AX = 0$  (always consistent, has trivial solution  $X = 0$ )
- **Non-homogeneous:**  $AX = B, B \neq 0$
- **Consistent:** Has solution(s)
- **Inconsistent:** No solution

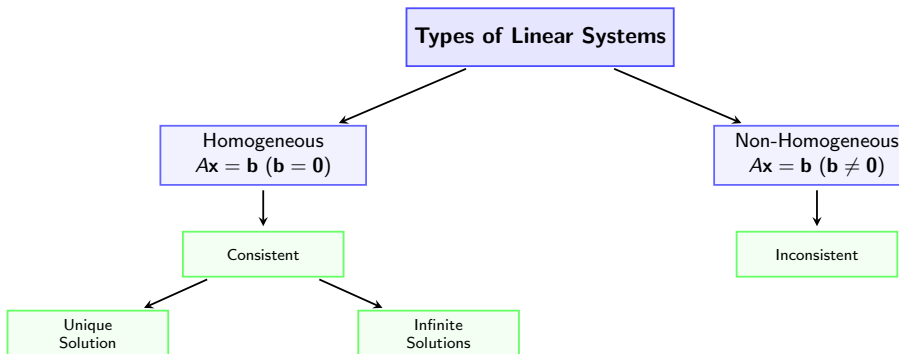
## Example systems:

$$1 \quad \begin{cases} 2x_1 + 5x_2 - x_3 = 5 \\ 3x_1 + 4x_2 + 2x_3 = 11 \\ x_1 + 2x_2 - 2x_3 = -3 \end{cases} \quad (\text{Consistent, unique solution})$$

$$2 \quad \begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 - x_2 + 7x_3 = 11 \\ 3x_1 + 5x_2 + 4x_3 = -3 \end{cases} \quad (\text{Consistent, infinite solutions})$$

$$3 \quad \begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 - 6x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 4x_3 = -3 \end{cases} \quad (\text{Inconsistent})$$

# Types of Linear Systems - Flow Chart



- **Homogeneous:** Always has at least trivial solution  $x = 0$
- **Non-homogeneous:** May be consistent or inconsistent

# Consistency Conditions

For  $AX = B$ :

Let  $r = \text{rank}(A)$ ,  $r' = \text{rank}([A|B])$ ,  $n = \text{number of variables}$ .

- 1 Consistent with unique solution:**  $r = r' = n$
- 2 Consistent with infinite solutions:**  $r = r' < n$
- 3 Inconsistent:**  $r \neq r'$

**Homogeneous system**  $AX = 0$ :

- Always consistent (trivial solution  $X = 0$ )
- Non-trivial solution exists iff  $|A| = 0$
- If  $|A| \neq 0$ , only trivial solution

# Solving Systems - Matrix Method

For  $AX = B$ , if  $|A| \neq 0$ :

$$X = A^{-1}B$$

**Example:** Solve 
$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ 2x_1 - 3x_2 + 2x_3 = 6 \\ 2x_1 + 2x_2 + x_3 = 5 \end{cases}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$|A| = 1(-3 - 4) - (-2)(2 - 4) + 1(4 + 6) = -7 - 4 + 10 = -1$$

$$A^{-1} = \frac{1}{-1} \text{adj}(A) = - \begin{bmatrix} -7 & 4 & -1 \\ 2 & -1 & 0 \\ 10 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 7 & -4 & 1 \\ -2 & 1 & 0 \\ -10 & 6 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution:  $x_1 = 1$ ,  $x_2 = 2$ ,  $x_3 = -1$

# Cramer's Rule

For  $AX = B$  with  $|A| \neq 0$ :

Let  $A_i$  = matrix  $A$  with column  $i$  replaced by  $B$ .

Then:

$$x_i = \frac{|A_i|}{|A|} \quad \text{for } i = 1, 2, \dots, n$$

**Example:** Solve  $\begin{cases} 3x_1 + x_2 - x_3 = -4 \\ x_1 + x_2 - 3x_3 = 4 \\ -x_1 + 2x_2 - x_3 = -1 \end{cases}$

$$|A| = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -3 \\ -1 & 2 & -1 \end{vmatrix} = 3(-1+6) - 1(-1-3) - 1(2+1) = 15 + 4 - 3 = 16$$

$$x_1 = \frac{\begin{vmatrix} -4 & 1 & -1 \\ 4 & 1 & -3 \\ -1 & 2 & -1 \end{vmatrix}}{16} = \frac{-4(-1+6) - 1(-4-3) - 1(8+1)}{16} = \frac{-20+7-9}{16} = \frac{-22}{16} = -\frac{11}{8}$$

Similarly:  $x_2 = \frac{8}{16} = \frac{1}{2}$ ,  $x_3 = \frac{18}{16} = \frac{9}{8}$

## Solving Systems - Row Reduction (Example 1)

$$\text{Solve } \begin{cases} 3x_1 + 2x_2 + x_3 = 3 \\ 4x_1 + 5x_2 - 3x_3 = -3 \\ 3x_1 - 2x_2 - 2x_3 = 17 \end{cases}$$

$$\text{Augmented matrix: } \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 4 & 5 & -3 & -3 \\ 3 & -2 & -2 & 17 \end{array} \right]$$

$$R_1 \leftrightarrow R_3: \left[ \begin{array}{ccc|c} 3 & -2 & -2 & 17 \\ 4 & 5 & -3 & -3 \\ 3 & 2 & 1 & 3 \end{array} \right]$$

$$R_2 \leftarrow R_2 - \frac{4}{3}R_1, R_3 \leftarrow R_3 - R_1: \left[ \begin{array}{ccc|c} 3 & -2 & -2 & 17 \\ 0 & \frac{23}{3} & -\frac{1}{3} & -\frac{77}{3} \\ 0 & 4 & 3 & -14 \end{array} \right]$$

Continue reducing to echelon form, then back substitute.

Final solution:  $x_1 = 3$ ,  $x_2 = -2$ ,  $x_3 = 1$

## Solving Systems - Row Reduction (Example 2)

$$\text{Solve } \begin{cases} x_1 - 2x_2 + 3x_3 = 4 \\ 2x_1 - 4x_2 + 6x_3 = 8 \\ -x_1 + 2x_2 - 3x_3 = -4 \end{cases}$$

$$\text{Augmented matrix: } \left[ \begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & -4 & 6 & 8 \\ -1 & 2 & -3 & -4 \end{array} \right]$$

Left as exercise...

# Homogeneous Systems - Non-trivial Solutions

For  $AX = 0$ :

Non-trivial solution exists iff  $|A| = 0$

**Example:** Find  $\lambda$  for non-trivial solution: 
$$\begin{cases} x + y + z = 0 \\ 2x + y - \lambda z = 0 \\ x + 2y - 2z = 0 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 1(-2 + 2\lambda) - 1(-4 + \lambda) + 1(4 - 1) =$$
$$-2 + 2\lambda + 4 - \lambda + 3 = \lambda + 5$$

For non-trivial solution:  $|A| = 0 \Rightarrow \lambda + 5 = 0 \Rightarrow \lambda = -5$

When  $\lambda = -5$ , solve: 
$$\begin{cases} x + y + z = 0 \\ 2x + y + 5z = 0 \\ x + 2y - 2z = 0 \end{cases}$$

Solution:  $x = -3t$ ,  $y = t$ ,  $z = t$  for any  $t \in \mathbb{R}$

# Summary

- Row reduction offers an efficient way to compute determinants for large matrices.
- Keep track of row operation effects:
  - Row multiplication by  $k$ : multiply determinant by  $k$ .
  - Row interchange: multiply determinant by  $-1$ .
  - Row addition: determinant unchanged.
- Column operations have analogous effects.
- Proportional rows/columns yield zero determinant.