LU Decomposition

Linear Algebra - 24 DS

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What is LU Decomposition?

Definition: LU decomposition factors a square matrix A into:

$$\boxed{A = LU}$$

where:

- L = **Lower triangular** matrix (with 1's on the diagonal)
- U = **Upper triangular** matrix

Why use LU decomposition?

1. Solve $Ax = b$ faster for multiple b vectors
2. Compute determinants easily: $\det(A) = \det(L) \cdot \det(U) = \prod u_{ii}$
3. Foundation for many numerical algorithms

Key idea: Elimination multipliers become entries of L.

Two Methods for LU Decomposition

Row Operations Method	**Classical Method (Formulas)**
Apply elimination to get U Track multipliers to build L L has 1's on diagonal	Use formulas: $u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$ $l_{ij} = \frac{1}{u_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right)$
Both methods give the same result!	

Note: Classical method is better for theoretical understanding.
Row operations method is better for hand calculation.

Example 1: LU Decomposition

Using Row Operations Method

Matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix}$$

Step 1: Start elimination to get U

Pivot: first entry $= 2$

Eliminate below it (make column 1 entries under pivot $= 0$):

$$R_2 \to R_2 - \frac{4}{2} \cdot R_1 = R_2 - 2R_1$$

$$R_3 \to R_3 - \frac{6}{2} \cdot R_1 = R_3 - 3R_1$$

$$U_1 = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 9 & 19 \end{bmatrix}$$

Example 1: Continue Elimination

Get U matrix

Step 2: Eliminate entry in R_3, column 2
Pivot: R_2, column 2 $= 1$

$$R_3 \to R_3 - \frac{9}{1} \cdot R_2 = R_3 - 9R_2$$

$$U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -26 \end{bmatrix}$$

Now U is **upper triangular**.

Step 3: Build L from multipliers
The multipliers we used during elimination become the entries of L:

- To eliminate (2,1): multiplier $= 4/2 = 2$
- To eliminate (3,1): multiplier $= 6/2 = 3$
- To eliminate (3,2): multiplier $= 9/1 = 9$

Example 1: Final LU Decomposition

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -26 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & -26 \end{bmatrix}}$$

Verification:

$$LU = \begin{bmatrix} 1 \cdot 2 & 1 \cdot 3 & 1 \cdot 1 \\ 2 \cdot 2 + 1 \cdot 0 & 2 \cdot 3 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 5 \\ 3 \cdot 2 + 9 \cdot 0 + 1 \cdot 0 & 3 \cdot 3 + 9 \cdot 1 + 1 \cdot 0 & 3 \cdot 1 + 9 \cdot 5 + 1 \cdot (-26) \end{bmatrix} = A$$

Matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 8 \end{bmatrix}$$

Step 1: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ (multipliers: 2, 3)

$$U_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

Step 2: $R_3 \to R_3 - 1 \cdot R_2$ (multiplier: 1)

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3: Build L:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Example 2: Final Result

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Verification:

$$LU = \begin{bmatrix} 1 & 2 & 3 \\ 2 \cdot 1 + 1 \cdot 0 & 2 \cdot 2 + 1 \cdot 1 & 2 \cdot 3 + 1 \cdot (-2) \\ 3 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 & 3 \cdot 2 + 1 \cdot 1 + 1 \cdot 0 & 3 \cdot 3 + 1 \cdot (-2) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 7 & 8 \end{bmatrix} = A$$

Example 3: When Pivot is Zero

PA = LU Decomposition (with Partial Pivoting)

Matrix:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Problem: First pivot is 0 $\rightarrow$ cannot eliminate.
Solution: Swap rows first (Partial Pivoting)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(swap } R_1 \text{ and } R_2\text{)}$$

$$PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Now apply LU decomposition to PA.

Example 3: Continue PA = LU

$$PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Step 1: $R_3 \to R_3 - \frac{7}{4} R_1$ (multiplier: 7/4)

$$U_1 = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & -\frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

Step 2: $R_3 \to R_3 - \left(-\frac{3/4}{2}\right) R_2 = R_3 + \frac{3}{8} R_2$

$$U = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{7}{4} & -\frac{3}{8} & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Classical Method (Doolittle Algorithm)

Formula-based approach

For $i = 1$ to n:

- **Step 1 (U row):** $u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad j = i, i+1, \ldots, n$
- **Step 2 (L column):** $l_{ji} = \frac{1}{u_{ii}} \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} u_{ki} \right), \quad j = i+1, \ldots, n$

Example: For $A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$

$u_{11} = a_{11} = 4$
$l_{21} = a_{21}/u_{11} = 6/4 = 1.5$
$u_{12} = a_{12} = 3$
$u_{22} = a_{22} - l_{21} u_{12} = 3 - (1.5)(3) = -1.5$

$$L = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix}$$

Example 4: Classical Method

3×3 Matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 3 \\ 9 & 8 & 6 \end{bmatrix}$$

Step 1: $u_{11} = 3,\ l_{21} = 6/3 = 2,\ l_{31} = 9/3 = 3$
Step 2: $u_{12} = a_{12} = 2,\ u_{13} = a_{13} = 1$
Step 3: $u_{22} = a_{22} - l_{21}u_{12} = 5 - (2)(2) = 1$
$u_{23} = a_{23} - l_{21}u_{13} = 3 - (2)(1) = 1$
Step 4: $l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}u_{12}) = \frac{1}{1}(8 - (3)(2)) = 2$
Step 5: $u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 6 - (3)(1) - (2)(1) = 1$

Example 4: Final Result

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 3 \\ 9 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

Check determinant:

$$\det(A) = \det(L) \cdot \det(U) = 1 \cdot (3 \times 1 \times 1) = 3$$

Comparison: Row Operations vs Classical Method

Row Operations Method	**Classical Method**
More intuitive	Better for programming
Direct visual feedback	Works without tracking multipliers
Good for hand calculations	Good for theoretical analysis
Can lose track of multipliers	More prone to arithmetic errors
When to use each: - Row operations: Teaching, small matrices - Classical: Computational implementation, large matrices	

Applications of LU Decomposition

1. **Solving linear systems:**

$$Ax = b \Rightarrow LUx = b$$

   Solve $Ly = b$ (forward substitution), then $Ux = y$ (back substitution)

2. **Computing determinants:**

$$\det(A) = \prod_{i=1}^{n} u_{ii}$$

3. **Matrix inversion:** Solve $AX = I$ efficiently
4. **Numerical stability:** With partial pivoting ($PA = LU$)

Complexity: $O(n^3)$ operations - same as Gaussian elimination, but reusability makes it faster for multiple RHS vectors.

Summary

$$\boxed{A = LU}$$

- L: Lower triangular with 1's on diagonal
- U: Upper triangular
- **Row operations method:** Track elimination multipliers
- **Classical method:** Use Doolittle formulas
- **Partial pivoting ($PA = LU$):** Handles zero pivots

Key takeaway: LU decomposition is Gaussian elimination in matrix form – a fundamental tool in linear algebra and numerical computing.