Introduction to Vector Spaces; Linear Combinations and Linear Independence

Linear Algebra - 24 DS

AH Sheikh

QUEST - Spring 2026

Why should we care about vectors?

Vectors are everywhere in CS:

- Machine Learning (feature vectors)
- Computer Graphics (3D coordinates)
- Data Science (high-dimensional data)
- Search engines (ranking vectors)
- Scientific Computing (state vectors)

Why should we care about vectors?

Vectors are everywhere in CS:

- Machine Learning (feature vectors)
- Computer Graphics (3D coordinates)
- Data Science (high-dimensional data)
- Search engines (ranking vectors)
- Scientific Computing (state vectors)

Key message: Linear algebra is the mathematics of data.

What is a Vector in CS?

In computer science, a vector is:

$$(v_1, v_2, \ldots, v_n) \quad or \quad [v_1, v_2, \ldots, v_n]$$

You can also write as

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Examples:

- RGB pixel: $(255, 100, 40)$
- Feature vector in ML
- Word embedding (300-dimensional vector)

What is a Vector in CS?

In computer science, a vector is:

$$(v_1, v_2, \ldots, v_n) \quad or \quad [v_1, v_2, \ldots, v_n]$$

You can also write as

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Examples:

- RGB pixel: $(255, 100, 40)$
- Feature vector in ML
- Word embedding (300-dimensional vector)

We will be interested in mathematical structure behind these objects(vectors).

Euclidean Vector Space $\mathbb{R}^n$

Definition:
A set of all n-tuples

$$\mathbb{R}^n = \{(v_1, \dots, v_n) \mid v_i \in \mathbb{R}\}$$

with two operations:
Addition (component-wise)

$$(u_1, \dots, u_n) + (v_1, \dots, v_n) = (u_1 + v_1, \dots, u_n + v_n)$$

Scalar multiplication

$$\alpha(v_1, \dots, v_n) = (\alpha v_1, \dots, \alpha v_n)$$

Example: Vector Operations

Let

$$u = (1, 2, 3), \quad v = (4, -1, 0)$$

Then:

$$u + v = (5, 1, 3)$$

$$3u = (3, 6, 9)$$

Example: Vector Operations

Let

$$u = (1, 2, 3), \quad v = (4, -1, 0)$$

Then:

$$u + v = (5, 1, 3)$$

$$3u = (3, 6, 9)$$

These operations satisfy:

- Associativity
- Commutativity
- Zero vector
- Additive inverse

This structure (set with operations and their properties) is called a **vector space**.

Linear Combinations (Core Concept)

Given vectors $v_1, \ldots, v_k$:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is called a **linear combination**.

Linear Combinations (Core Concept)

Given vectors $v_1, \ldots, v_k$:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is called a **linear combination**.
Example:

$$v_1 = (1, 0), \quad v_2 = (0, 1)$$

Then:

$$3v_1 + 2v_2 = (3, 2)$$

Linear Combinations (Core Concept)

Given vectors $v_1, \ldots, v_k$:

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

is called a **linear combination**.
Example:

$$v_1 = (1, 0), \quad v_2 = (0, 1)$$

Then:

$$3v_1 + 2v_2 = (3, 2)$$

Everything in linear algebra revolves around linear combinations.

Span: Generative Power

Definition:

$$\text{span}\{v_1, \ldots, v_k\} = \{a_1 v_1 + \cdots + a_k v_k\}$$

Span: Generative Power

Definition:

$$\text{span}\{v_1, \ldots, v_k\} = \{a_1 v_1 + \cdots + a_k v_k\}$$

Example 1:
If $v = (1, 2)$, then

$$\text{span}\{v\}$$

is a line through the origin.

Span: Generative Power

Definition:

$$\text{span}\{v_1, \ldots, v_k\} = \{a_1 v_1 + \cdots + a_k v_k\}$$

Example 1:
If $v = (1, 2)$, then

$$\text{span}\{v\}$$

is a line through the origin.
Example 2:

$$v_1 = (1, 0), \quad v_2 = (0, 1)$$

$$\text{span}\{v_1, v_2\} = \mathbb{R}^2$$

Linear Independence

Vectors $v_1, \ldots, v_k$ are linearly independent if

$$a_1 v_1 + \cdots + a_k v_k = 0$$

implies

$$a_1 = \cdots = a_k = 0$$

Linear Independence

Vectors $v_1, \ldots, v_k$ are linearly independent if

$$a_1 v_1 + \cdots + a_k v_k = 0$$

implies

$$a_1 = \cdots = a_k = 0$$

Example:

$$(1, 0), (0, 1) \text{ are independent}$$

$$(1, 2), (2, 4) \text{ are dependent}$$

Why? One is a scalar multiple of the other.

Basis and Dimension

A **basis** is:

- Linearly independent
- Spans the space

Basis and Dimension

A **basis** is:

- Linearly independent
- Spans the space

Examples:

$$\{(1,0),(0,1)\}$$

is a basis of $\mathbb{R}^2$.

Basis and Dimension

A **basis** is:

- Linearly independent
- Spans the space

Examples:

$$\{(1,0),(0,1)\}$$

is a basis of $\mathbb{R}^2$.
Dimension = number of basis vectors.

$$\dim(\mathbb{R}^n) = n$$

Basis and Dimension

A **basis** is:

- Linearly independent
- Spans the space

Examples:

$$\{(1,0),(0,1)\}$$

is a basis of $\mathbb{R}^2$.
Dimension = number of basis vectors.

$$\dim(\mathbb{R}^n) = n$$

Interpretation: degrees of freedom.