Linear Combinations and Spanning

Linear Algebra - 24 DS

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Quick Recap

From the previous lecture

- **Vector:** An n-tuple $(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$
- **Linear combination:** Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

- **Span:** Set of **all** linear combinations

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \left\{ \sum_{i=1}^{k} c_i\mathbf{v}_i \;\middle|\; c_i \in \mathbb{R} \right\}$$

Today: Lots of examples and practice!

Example 1: Simple Linear Combination

$\mathbb{R}^2$

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Compute: $3\mathbf{v}_1 - 2\mathbf{v}_2$

$$3\mathbf{v}_1 - 2\mathbf{v}_2 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

$$\boxed{3\mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 8 \\ -3 \end{bmatrix}}$$

Example 2: Writing a Vector as Linear Combination

$\mathbb{R}^2$

Write $\mathbf{w} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Step 1: Set up $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = 5 \\ 2c_1 + c_2 = 4 \end{cases}$$

Step 2: Solve: From first, $c_1 = 5 - 2c_2$
Substitute: $2(5 - 2c_2) + c_2 = 10 - 4c_2 + c_2 = 10 - 3c_2 = 4$
$\Rightarrow 3c_2 = 6 \Rightarrow c_2 = 2$, then $c_1 = 5 - 4 = 1$

$\boxed{\mathbf{w} = 1\mathbf{v}_1 + 2\mathbf{v}_2}$

Example 3: When Linear Combination Fails

$\mathbb{R}^2$

Question: Can $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ be written as LC of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Observation: $\mathbf{v}_2 = 2\mathbf{v}_1$ (vectors are collinear)

Attempt: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = (c_1 + 2c_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$

Any LC gives a vector with equal components!

Since $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has different components:

$$2 \neq 3 \quad \Rightarrow \quad \text{Impossible!}$$

Example 4: Span of a Single Vector

$\mathbb{R}^2$ and $\mathbb{R}^3$

In $\mathbb{R}^2$:

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{span}\{\mathbf{v}\} = \left\{ t \begin{bmatrix} 2 \\ -1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

Geometric meaning: **Line through origin** with direction $(2, -1)$

In $\mathbb{R}^3$:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{span}\{\mathbf{v}\} = \{(t, 2t, 3t) \mid t \in \mathbb{R}\}$$

Still a **line through origin** in 3D!

Example 5: Span of Two Vectors in $\mathbb{R}^3$

A plane through origin

Vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_1 + c_2 \end{bmatrix} \,\middle|\, c_1, c_2 \in \mathbb{R} \right\}$$

Geometric meaning: A **plane through origin** in $\mathbb{R}^3$

Equation of this plane: $x + y - z = 0$ (Check:

Example 6: Span of Three Vectors

Can they span all of $\mathbb{R}^3$?

Vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Standard basis! $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$

What if:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

All are multiples of $(1, 2, 3)$ span is a **line** (dimension 1)

What if:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdots \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ \cdots \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ \cdots \end{bmatrix}$$

Example 7: Do These Span $\mathbb{R}^2$?

Question: Does $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Check: $\mathbf{v}_2 = 2\mathbf{v}_1$ Vectors are collinear

Span is a **line**, not the whole plane.

$$\boxed{\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq \mathbb{R}^2}$$

When do two vectors span $\mathbb{R}^2$? When they are NOT multiples of each other (linearly independent)

Example 8: Geometric Interpretation

Visualizing spans in $\mathbb{R}^2$ and $\mathbb{R}^3$

Set of vectors	**Span**
One non-zero vector	Line through origin
Two non-collinear vectors in $\mathbb{R}^2$	Entire $\mathbb{R}^2$ (plane)
Two non-collinear vectors in $\mathbb{R}^3$	Plane through origin
Three non-coplanar vectors in $\mathbb{R}^3$	Entire $\mathbb{R}^3$

Key insight: Span = all points reachable by linear combinations = a **subspace** (line, plane, or whole space)

Example 9: Span in Higher Dimensions

$\mathbb{R}^4$

Vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

This is a **3-dimensional subspace** of $\mathbb{R}^4$ (a hyperplane).

Important: Span can have lower dimension than the ambient space!

Problem 1: Is $\mathbf{w}$ in the Span?

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$$

Question: Is $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

Method: Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$:

$$\begin{cases} c_1 + 2c_2 = 4 \\ -c_1 + c_2 = -1 \\ 2c_1 + 3c_2 = 7 \end{cases}$$

From first two: Add equations $\rightarrow 3c_2 = 3 \Rightarrow c_2 = 1$, then $c_1 = 2$
Check third: $2(2) + 3(1) = 4 + 3 = 7$

$$\boxed{\mathbf{w} = 2\mathbf{v}_1 + 1\mathbf{v}_2 \quad \Rightarrow \quad \text{YES}}$$

Problem 2: Is $\mathbf{w}$ in the Span?

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

Observation: $\mathbf{v}_2 = 2\mathbf{v}_1$ both vectors lie on same line
Span $= \text{span}\{\mathbf{v}_1\} = \{(t, 2t, -t) \mid t \in \mathbb{R}\}$

For $\mathbf{w} = (3, 6, 0)$, we need:

$$(t, 2t, -t) = (3, 6, 0) \Rightarrow t = 3 \text{ and } -t = 0 \Rightarrow t = 0$$

Contradiction!

$\boxed{\mathbf{w} \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}}$

Problem 3: Condition for Being in Span

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Question: What condition must $\mathbf{w} = (a, b, c)$ satisfy to be in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

Method: Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$:

$$\begin{cases} c_1 + 2c_2 = a \\ 2c_1 + c_2 = b \\ c_1 + 2c_2 = c \end{cases}$$

From first and third: $a = c$ This is the condition!

$$\boxed{\mathbf{w} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \iff a = c}$$

Problem 4: Span of Three Vectors

Given:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Question: Does $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$?

Check: Are they independent? Try $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$:

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

Add all three: $2(c_1 + c_2 + c_3) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$
From first: $c_2 = -c_1$, second: $c_3 = -c_1$
Third: $(-c_1) + (-c_1) = -2c_1 = 0 \Rightarrow c_1 = 0$ all zero independent!

Problem 5: Is This a Spanning Set?

Question: Does span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Observation: All vectors are multiples of $(1, 2)$
Span is the **line** $y = 2x$, not the whole plane.

No. Span is a 1-dimensional line in $\mathbb{R}^2$

Key point: Having more vectors doesn't guarantee larger span!
Redundant vectors don't add new directions.

Summary: Linear Combinations & Spanning

Concept	**Key Idea**
Linear Combination	$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$
Span	Set of **all** linear combinations
Geometric meaning	Line (1 vec), Plane (2 indep vecs), Space (3 indep vecs)
Checking if $\mathbf{w} \in$ span	Solve linear system
Redundant vectors	Don't change the span

Span = "All vectors you can reach" = A subspace!

Practice Problems

1. Write $\mathbf{w} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ as LC of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
2. Does span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\} = \mathbb{R}^3$?
3. Find condition on (a, b, c) so that $\mathbf{w} = (a, b, c)$ is in span of $\mathbf{v}_1 = (1, 1, 1), \mathbf{v}_2 = (1, 1, 0)$.
4. What is the geometric shape of span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$?
5. Can $\mathbf{w} = (0, 0, 1)$ be written as LC of $(1, 0, 0)$ and $(0, 1, 0)$? Why or why not?

Remember: Span = All possible destinations using given directions!