Subspaces and Linear Independence

Linear Algebra - 24 DS

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QUEST - Spring 2026

Outline

Definition of a Subspace

Definition

A non-empty subset W of $\mathbb{R}^n$ is called a **subspac** if it is closed under:

1. **Vector addition:** For all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$
2. **Scalar multiplication:** For all $\mathbf{u} \in W$ and $c \in \mathbb{R}$, $c\mathbf{u} \in W$

Important Consequence of Subspace Definition

Zero Vector Requirement

Every subspace must contain the zero vector $\mathbf{0}$!

Proof.

Since W is non-empty, choose any $\mathbf{u} \in W$.

- By scalar multiplication property, for $c = 0$:
- $0 \cdot \mathbf{u} = \mathbf{0} \in W$

□

Quick Check

Is $W = \{(x, y) \in \mathbb{R}^2 : y = x + 1\}$ a subspace?

- Contains $(0, 1)$ but not $(0, 0)$
- Therefore, **not** a subspace!

Examples of Subspaces in $\mathbb{R}^2$

Lines through origin

$W = \{t\mathbf{v} : t \in \mathbb{R}\}$ for any fixed $\mathbf{v} \neq \mathbf{0}$

- x-axis: $\{(x, 0) : x \in \mathbb{R}\}$
- y-axis: $\{(0, y) : y \in \mathbb{R}\}$
- Line $y = 2x$: $\{(t, 2t) : t \in \mathbb{R}\}$

Trivial subspaces

- $\{\mathbf{0}\}$ - zero subspace
- $\mathbb{R}^2$ itself

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Non-examples

- Lines not through origin
- First quadrant
- Circles or curves

Examples of Subspaces in $\mathbb{R}^3$

Common subspaces in $\mathbb{R}^3$

- **Lines through origin:** $\{t\mathbf{v} : t \in \mathbb{R}\}$
- **Planes through origin:** $\{s\mathbf{u} + t\mathbf{v} : s, t \in \mathbb{R}\}$ for independent $\mathbf{u}, \mathbf{v}$

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Specific examples:

- x-axis: $\{(t, 0, 0)\}$
- xy-plane: $\{(x, y, 0)\}$
- Plane $z = 0$
- Plane $x + y + z = 0$

Non-examples:

- Plane $z = 1$
- Unit sphere
- First octant

Visualizing Subspaces: Plane $x + y + z = 0$ in $\mathbb{R}^3$

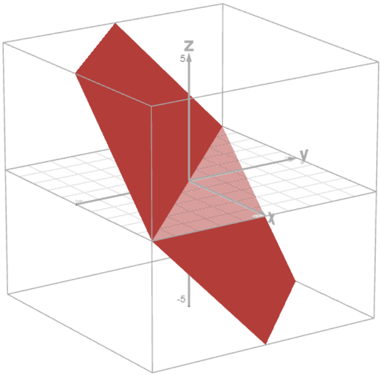

Figure: Plane $x + y + z = 0$

Visualizing Subspaces: Plane $x + y + z = 0$ in $\mathbb{R}^3$

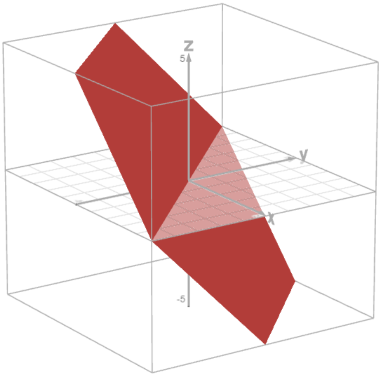

Figure: Plane $x + y + z = 0$

Visualizing Subspaces: Plane $x + y + z = 0$ in $\mathbb{R}^3$

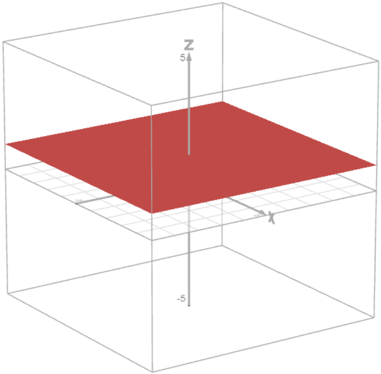

Figure: Plane $z = 1$

Subspace Example 1

The xy-plane (All vectors with $z = 0$)

Set:

$$W_1 = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

Check subspace conditions:

1. **Zero vector:** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W_1 \quad (\text{take } x = 0, y = 0)$

2. **Closed under addition:**

$$\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} \in W_1$$

Subspace Example 2

Line through origin along vector $(1,1,1)$

Set:

$$W_2 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

Check subspace conditions:

1. **Zero vector:** $t = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W_2$
2. **Closed under addition:**

$$\mathbf{u} = \begin{bmatrix} t_1 \\ t_1 \\ t_1 \end{bmatrix}, \; \mathbf{v} = \begin{bmatrix} t_2 \\ t_2 \\ t_2 \end{bmatrix} \in W_2$$

Non-Example: What is NOT a Subspace?

The union of two lines through origin

Set:

$$W_3 = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}$$

(x-axis y-axis in the xy-plane)

Why it fails:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in W_3, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in W_3$$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin W_3$$

Definition of Linear Independence

Definition (Standard Definition)

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are **linearly independent** if the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_k = 0$.

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Interpretation

In $\mathbb{R}^2$: Two vectors are independent iff they're not collinear.
In $\mathbb{R}^3$: Three vectors are independent iff they're not coplanar.

How to Check Linear Independence

[Matrix Method] To check if vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are independent:

1. Form matrix A with vectors as *columns*
2. Find rank(A) (or RREF)
3. If rank(A) $= k$, vectors are independent
4. If rank(A) $< k$, vectors are dependent

[Determinant Method for $n \times n$ matrices] For n vectors in $\mathbb{R}^n$:

- Form $n \times n$ matrix with vectors as columns
- Compute determinant
- $\det \neq 0 \iff$ vectors are independent

Example 1: $\mathbb{R}^2$ Independence Check

Example

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Solution:

Method 1: Form matrix and find determinant

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$$\det(A) = (1)(4) - (3)(2) = 4 - 6 = -2 \neq 0$$

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Conclusion

Since $\det(A) \neq 0$, the vectors are **linearly independent**.

Example 2: $\mathbb{R}^3$ Independence Check

Example

Determine if $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 1, 3)$, $\mathbf{v}_3 = (1, 1, 2)$ are independent.

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Conclusion

$\text{rank}(A) = 2 < 3 \implies$ vectors are **linearly dependent**.

Note: $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$

Practice Problem 1

Problem

For what value(s) of h are the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (2, 4, 6)$, and $\mathbf{v}_3 = (1, h, 5)$ linearly dependent?

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Key Insight

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are already dependent, any set containing them is dependent regardless of $\mathbf{v}_3$!

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Prove that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly independent in* $\mathbb{R}^3$*, then* $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ *is also linearly independent.*

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- Orthogonal to $(1,1,0)$: $x+y=0 \implies y=-x$

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Answer

All vectors of the form $t(1, -1, -1)$ where $t \in \mathbb{R}$. This is a line through the origin (a 1-dimensional subspace).