LECTURE 1

SETS

What is a Set?

- A *set* is a well-defined collection of distinct objects.
  - No duplicates
  - Order does not matter
- The objects in a set are called the *elements* or *members* of the set.
- Capital letters $A,B,C,\ldots$ usually denote sets.
- Lowercase letters $a,b,c,\ldots$ denote the elements of a set.

Examples

- The collection of the vowels in the word “probability”.
- The collection of students’ IDs in this class.
- The collection of two-digit positive integers divisible by 5.
- The collection of great football players in the National Football League.
- The collection of intelligent members of the United States Congress.

Mathematical Examples

- {2,3,5,7,11}
- {(1,1), (2,2), (3,3)}
- {Apple, Orange, Banana, Peach}
- {Apple, Dell, IBM}
- {Heads, Tails}
- {Win, Lose, Tie}
- {}

No Duplicates

- {1,1,2,3,5,8} is not a set
- {1,2,3,5,8} is a set

Order does not matter

- {3,4} = {4,3}
- {1,2,3,4,5} = {2,3,4,5,1}
- {1,4,2,5,3} = {1,3,5,2,4}
- {Apple, Dell, IBM} = {Dell, Apple, IBM}

The Empty Set

- The set with no elements.
- Also called *the <u>null set</u>*.
- Denoted by the symbol ϕ.
- Example: The set of real numbers *x* that satisfy the equation

$$2x + 1 = 3$$

Finite and Infinite Sets

- A finite set is one which can be counted.
- Example: The set of two-digit positive integers has 90 elements.
- An infinite set is one which cannot be counted.
- Example: The set of students' ID in this class.

The Cardinality of a Set

- Simple; **size of Set**
- Notation: $n(A)$
- For finite sets A, $n(A)$ is the number of elements of A.
- For infinite sets A, write $n(A)=\infty$.

Specifying a Set

- List the elements explicitly, e.g.,

$$C = \{ a, o, i \}$$

- List the elements implicitly, e.g.,

$$K = \{ 10, 15, 20, 25, ..., 95 \}$$

- Use set builder notation, e.g.,

$$Q = \{ x \mid x = p/q \quad \text{where } p \text{ and } q \text{ are integers and } q \neq 0 \}$$

The Universal Set

- A set *U* that includes all of the elements under consideration in a particular discussion or problem.
- Depends on the context.
- Examples: The set of Latin letters, the set of natural numbers, the set of points on a line.

The Membership Relation

- Let A be a set and let x be some object.
- Notation: $x \in A$
- Meaning: *x* is a member of *A*, or *x* is an element of *A*, or *x* belongs to *A*.
- Negated by writing $x \notin A$
- Example: $V = \{a, e, i, o, u\}$. $e \in V$, $b \notin V$.

Element-of Notation

- "$x \in S$" means that x is an element of the set S.
- $1 \in \{1,2,3\}$
- $2 \in \{1,2,3\}$
- $3 \in \{1,2,3\}$

Not-an-element-of Notation

- "$x \notin S$" means that x is **not** an element of the set S.
- $0 \notin \{1,2,3\}$
- $4 \notin \{1,2,3\}$
- $17 \notin \{1,2,3\}$

Equality of Sets

- Two sets *A* and *B* are equal, denoted $A=B$, if they have the same elements.
- Otherwise, $A \neq B$.
- Example: The set *A* of odd positive integers is not equal to the set *B* of prime numbers.
- Example: The set of odd integers between 4 and 8 is equal to the set of prime numbers between 4 and 8.

Subsets

- *A* is a subset of *B* if every element of *A* is an element of *B*.
- Notation: $A \subseteq B$
- For each set *A*, $A \subseteq A$
- For each set *B*, $\emptyset \subseteq B$
- *A* is proper subset of *B* if $A \subseteq B$ and $A \neq B$

The *Power Set* Operation

- The *power set* P(*S*) of a set *S* is the set of all subsets of *S*. P(*S*) = $\{x \mid x \subseteq S\}$.
- *E.g.* P({a,b}) = {∅, {a}, {b}, {a,b}}.
- Sometimes P(*S*) is written $\mathbf{2}^S$. Note that for finite *S*, $|P(S)| = 2^{|S|}$.
- It turns out that |P(**N**)| > |**N**|. *There are different sizes of infinite sets*!

Example 1

- Set A = {1, 2}
- P(A) = { ∅, {1}, {2}, {1,2} }

Example 2

- Set B = {a, b, c}
- P(B) = { ∅, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

Example 3

- Set C = {x, y, z, w}
- P(C) has 16 subsets:
- { ∅, {x}, {y}, {z}, {w}, {x,y}, {x,z}, {x,w}, {y,z}, {y,w}, {z,w}, {x,y,z}, {x,y,w}, {x,z,w}, {y,z,w}, {x,y,z,w} }

Set Operations

- Sets are mathematical objects which conforms different operations.

Few are following…

Unions

- The union of two sets *A* and *B* is

$$A \cup B = \left\{ x \mid x \in A \text{ or } x \in B \right\}$$

- The word "or" is inclusive.

Example 1

- $A = \{1, 2, 3\}$
- $B = \{3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5\}$

Example 2

- $X = \{a, b\}$
- $Y = \{b, c, d\}$
- $X \cup Y = \{a, b, c, d\}$

Example 3

- $P = \{2, 4, 6, 8\}$
- $Q = \{1, 2, 3, 4\}$
- $P \cup Q = \{1, 2, 3, 4, 6, 8\}$

Intersections

- The intersection of *A* and *B* is

$$A \cap B = \left\{ x \mid x \in A \text{ and } x \in B \right\}$$

- Example: Let *A* be the set of even positive integers and *B* the set of prime positive integers. Then

$$A \cap B = \{2\}$$

- Definition: *A* and *B* are disjoint if

$$A \cap B = \emptyset$$

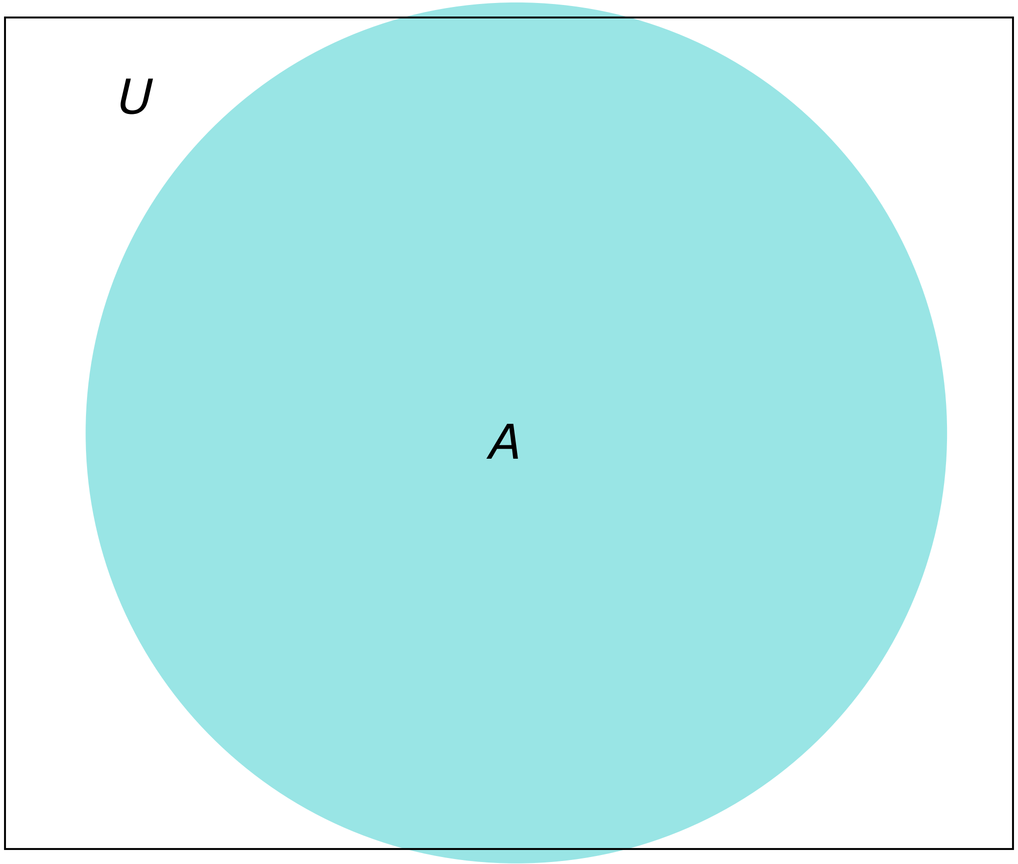
Complements

- If *A* is a subset of the universal set *U*, then the complement of *A* is the set

$$A^c = \left\{ x \in U \;\middle|\; x \notin A \right\}$$

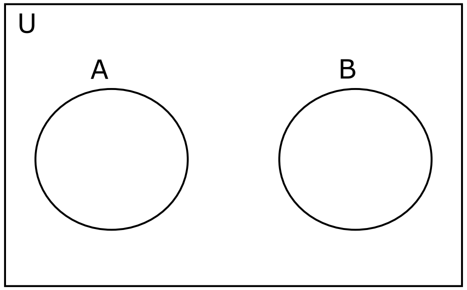
- Note: $A \cap A^c = \Phi$; $A \cup A^c = U$

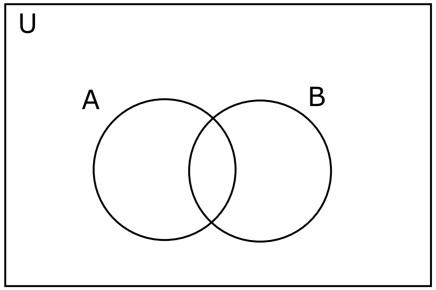
Venn Diagrams

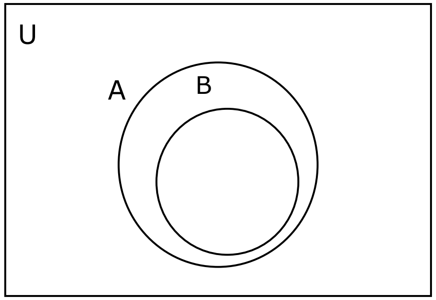


Set *A* represented as a disk inside a rectangular region representing *U.*

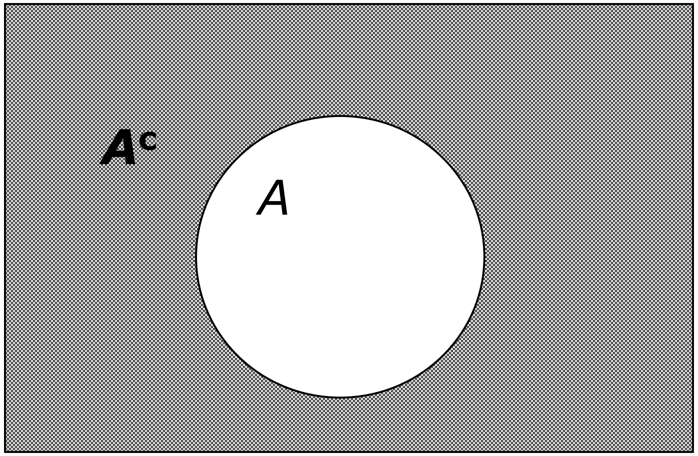
Possible Venn Diagrams for Two Sets





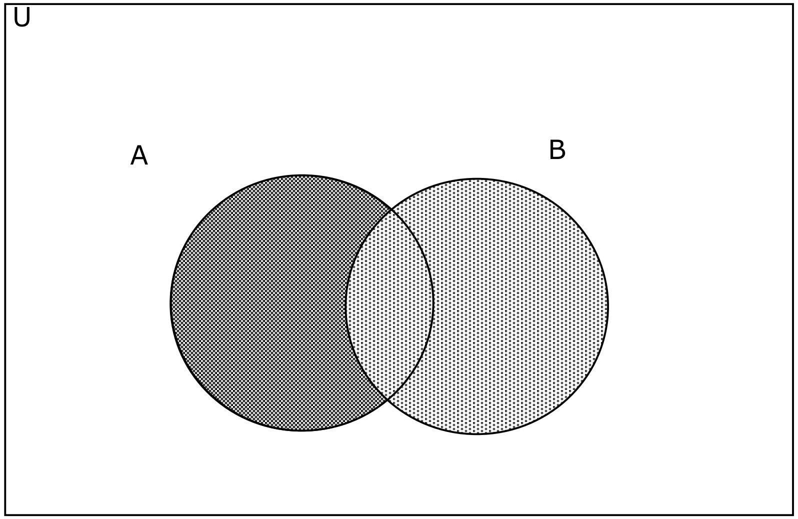


The Complement of a Set

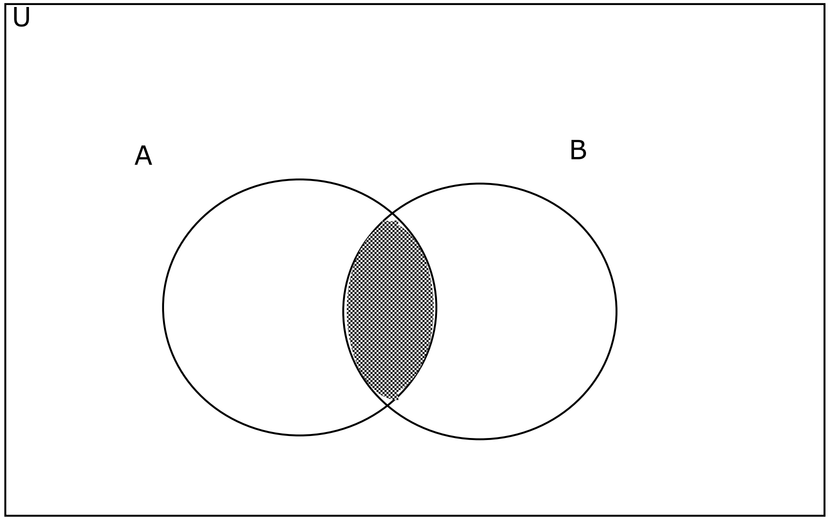


The shaded region represents the complement of the set *A*

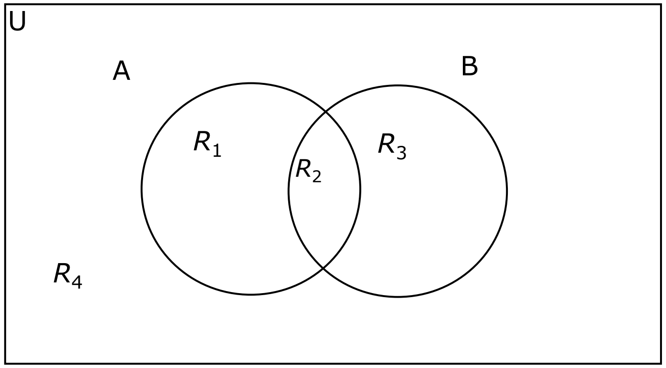
The Union of Two Sets



The Intersection of Two Sets



Sets Formed by Two Sets



- $R_1 = A \cap B^c$
- $R_2 = A \cap B$
- $R_3 = A^c \cap B$
- $R_4 = A^c \cap B^c$

Set Identities

- Identity: $A \cup \varnothing = A \quad A \cap U = A$
- Domination: $A \cup U = U \quad A \cap \varnothing = \varnothing$
- Idempotent: $A \cup A = A = A \cap A$
- Double complement: $\overline{(\overline{A})} = A$
- Commutative: $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Identities

To prove statements about sets, of the form $E_1 = E_2$ (where *E*s are set expressions), here are three useful techniques:

- Prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.
- Use logical equivalences.
- Use a *membership table*.

Method 1: Mutual subsets

Example: Show

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
  - Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$.
  - We know that $x \in A$, and either $x \in B$ or $x \in C$.
    - Case 1: $x \in B$. Then $x \in A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.
    - Case 2: $x \in C$. Then $x \in A \cap C$, so $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $x \in (A \cap B) \cup (A \cap C)$.
  - Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
- Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. …

Method 3: Membership Tables

- Just like truth tables for propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use "1" to indicate membership in the derived set, "0" for non-membership.
- Prove equivalence with identical columns.

Membership Table Example

Prove $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Membership Table Exercise

Prove $(A \cup B) - C = (A - C) \cup (B - C)$.

A	B	C	$A \cup B$	$(A \cup B) - C$	$A - C$	$B - C$	$(A - C) \cup (B - C)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Two Basic Counting Rules

If *A* and *B* are finite sets,

1. $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

2. $$n(A \cap B^c) = n(A) - n(A \cap B)$$

See the preceding Venn diagram.